ON THE ROLLING OF SHIPS.

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I feel some diffidence in bringing before the experienced members of this society what assumes to be a tolerably complete theoretical elucidation of a difficult and intricate subject, which has hitherto been treated as if unapproachable by the methods of regular investigation.

I may, however, perhaps, bespeak some attention to it, by mentioning that it is the result of an inquiry undertaken at the request of the late Mr. Brunel, with whom I frequently discussed its fundamental principles, while he was engaged on the design and the construction of the *Great Eastern*, receiving, as no one could fail to receive who discussed such principles with him, the greatest assistance from his broad and masculine perception of their real bearings and of their mutual relations.

I will add that having, in connection with the subject, undertaken to work out the probable seagoing properties of the ship herself, I had the good fortune to be associated with a member of his regular engineering staff, Mr. W. Bell, a personal friend of my own, an able mathematician and an accurate calculator, who, I am bound to say, not only relieved me of all the laborious part of the detail numerical calculations relating to the ship, but joined so heartily and effectually in the theoretical part of the investigation that there are very few parts which I am inclined to call specially my own.

The most observable feature in the actual movements of a ship when rolling, and that which had always appeared to me to be specially characteristic of the dynamical laws to which it would be necessary to refer them, is the gradual accumulation of angle during several successive rolls; the cumulative action thus growing up into a maximum, and

* As the substance of this Paper was placed before the Institution in an extemporary address, the Author has since been good enough to revise and extend it somewhat, in order to give it completeness as nearly as possible.—Ed.

then dying out by very similar gradations, until the ship becomes for a moment steady, when a nearly similar series of excursions commences and is reproduced: while in reference to the momentary pause, or cessation of motion, it has seemed to me clear that it occurs, not because the waves themselves cease, or cease to act, but because the last oscillation has died out at a moment when the ship and the waves have come to occupy, relatively, a position of momentary equilibrium.

It is not indeed easy to obtain by one's own observation, or to collect from others, such a complete series of facts as would enable one to arrive empirically at the rules by which these characteristic features of the operation are governed, and the existence of which they suggest; or even such as would justify the positive assertion that when tested in really heavy weather the behaviour of a ship can be compared, by a true analogy, with that which I have said to be her characteristic behaviour under ordinary circumstances. For the scenery, so to call it, which surrounds the phenomena of rolling motion (especially when these are developed on a very large scale), is for the most part so very striking, and appeals so forcibly to the imagination, that it is almost impossible for a landsman to divest his appreciation of them from passionate colouring, even though he applies himself to the task with the most determinately prosaic intentions. While even those whose life is spent among such phenomena, and who have become familiarized to them by habit, have become accustomed chiefly to regard them under their impressive aspect, and to mould on this all the phraseology in which they describe them; so that even from nautical men it is not easy to obtain statements which can safely be reduced to measure and number.

The best information, however, which I have been able to collect from the report of others, and from my own observation, confirms me in the belief that the very large angles of rolling which are occasionally reached are never due to single wave impulses, but are invariably the cumulative results of the operation of successive waves. And I believe, too, that the law of accumulation does, in fact, accord very closely with that which is arrived at in the following investigation.

This aspect of the question is so closely analogous to what happens when any oscillating body, such as a pendulum, is subjected to a series of impulses, partially synchronous with its own excursions, that it had always seemed to me probable that the laws which govern the latter class of phenomena, would be found, *mutatis mutandis*, applicable to the elucidation of the former also; and in attempting to investigate regularly on this line of thought the dynamical relations of a ship, and of the waves on which she floats, it turned out that the solution was less difficult than had been expected, and that its fundamental results, at least, could be arrived at with considerable completeness and closeness of approximation.

The investigation, then, of the laws of rolling motion in ships when thus regarded, assumes the form of the inquiry, "What is the cumulative result of the continuous action of "a series of consecutive waves operating on a given ship?" And in order to determine this, it is necessary first to determine how each individual wave will act on her at each instant of time. "What attitude does the ship, at each instant, seek to occupy in reference "to the wave on which she floats, and what measures the force which urges her to take "that attitude, or, to speak more strictly, what is the *position of momentary equilibrium* for "a body floating on a wave, and what accelerating force towards that position will the "body experience in terms of her momentary deviation from it?"

The answer to this question obviously depends on the nature and direction of the special forces which wave motion develops. And though a complete solution of this problem is no doubt profoundly difficult, and will not here be attempted, yet we have within our reach certain plain fundamental conditions which are strictly and practically applicable to our purpose.

It must not be considered a mere empty truism when it is remarked that the characteristic difference between still water and undulating water is, that while in the former *the particles are stationary and the surface is horizontal*, in the latter the *particles are in motion and the surface inclined.* For it is in the precise relation which the inclination of the surface bears to the motion of the particles, and again in the analogy of this relation to that under which a still surface is also a horizontal surface, that we shall find a key to the required solution.

When a fluid is at rest, the effort or "action," of a particle at the surface, consists solely of its own gravitation; and the direction of this is simply perpendicular and downwards. *Vice versâ*, the "reaction" by which the particle is supported and kept at rest is the resultant of the derivative pressures of the contiguous particles; and this reaction, since it precisely supports the surface particle and keeps it at rest, is, of course, precisely equal to its downward action, and in the opposite direction — that, is to say, perpendicular and upwards.

When the fluid is in motion the continual change, observed experimentally, in the motion of a surface particle (upwards and downwards, backwards and forwards), implies that it is subject to corresponding accelerating forces: hence the "action" of the particle is no longer that of simple gravitation, but that of gravitation compounded in every instance with the corresponding accelerating force which governs its path; and this composition of forces produces an appropriate inclined resultant, the direction and magnitude of which express precisely the whole action of the particle to which it belongs. *Vice versâ*, here also, as in still water, the corresponding "reaction" is the resultant of the derivative pressure of the contiguous particles, and this must be equal and in an opposite direction to the action which it counterbalances—the inclined resultant, namely, of the gravity of the particle and of the accelerated force which governs its path.

Now, in order that the resultant of all the pressures on a given particle at the surface of a fluid should lie in a particular line, it is necessary that the surface itself should be at right angles to that line. For, otherwise, there would be a preponderance of lateral pressure with reference to the line—a condition at variance with the condition that it is the true resultant. Thus it is that the resultant in still water being of necessity exactly perpendicular, the surface is of necessity exactly level; and thus it will also follow that in undulating water, the resultant having that degree of obliquity which the motions of the particles prescribe, the surface must of necessity be exactly as much "out of level" as the resultant is "out of perpendicular"—the *slope* or steepness of the surface gauges the *slant* or inclination of the resultant.

The Rationale, so to call it, and the practical result of this proposition, will be perhaps more readily understood by reflecting on the conditions to which we must adhere if we would give motion to a board on which a marble, or an adjusted spirit level, rests; or, again, to a cup brimful of liquid, without displacing the marble, or disturbing the level, or spilling the liquid. It will be felt at once that the surface of the board, or the plane of the cup's edge, must be "canted;" the degree of "cant" depending not on the the velocity of the motion, but on the changes of the velocity: and that, in fact, this angle must be determined by exactly those conditions which I have pointed out as governing the slope of a wave. The cup full of liquid, thus carried, is in fact equivalent to a small aggregation of moving particles, scooped out of, or detached from, the side of some kind of wave. But perhaps the simplest and readiest illustration of the principle will be obtained by attaching any one of these tests of level to a pendulum, at the centre of oscillation, on a plane at right angles to the suspension-rod. When the pendulum is allowed to oscillate under the influence of gravity, the plane of equilibrium will continue always at right angles to the radius of suspension; and the various tests of level will continue to occupy that plane without disturbance.

It follows, if the analogy to wave motion has been correctly stated, that either of these tests may be applied with success to actual wave surfaces. And, in point of fact, it is not a paradox to say that a properly-constructed float, which would carry a marble, or a bullet, in barely stable equilibrium when floating level on still water, would carry it also without disturbance when floating inclined on the steepest wave slope. This, indeed, would be a somewhat delicate experiment to try. A friend of mine has, however, verified the proposition with a floating spirit level: and I myself have verified it as follows:—A float was formed of cork, somewhat like a small life-buoy, about four inches in diameter; a mast was planted obliquely in one side of it, with its apex perpendicular over the centre of the float; a small plumb-bob was suspended from this, having its centre at the level of the centre of buoyancy of the float, and occupying, when in still water, the centre of the ring. When this was set afloat in a trough, fitted with apparatus for generating waves, while the plane of its flotation followed the slope of the waves, the plumb-bob remained, nevertheless, so completely central, that to an eye resting on it, it was difficult to believe that the surface was really disturbed by waves, though on watching the sides of the trough, it was plain that the wave slope ranged up to 15° or 20°; the plumb-line, at the same time, deviating to the same extent from the perpendicular. And on trying the experiment in the sea, I have seen the bob remain equally central, while the float rested in the hollow front of a breaking surf wave, even where the surface was considerably "over-hanging," so that the line sloped upward from the point of suspension.

It is, then, rigorously true, that to a particle at any point in the surface of a wave, or of any non-horizontal volume of free fluid, a tangent to the surface at the point in question is virtually level: and the same law will hold good if for a particle of fluid we substitute a particle of matter which floats, and which, therefore, accepts all the dynamical conditions which the position imposes on it. And if the configuration or character of the substituted particle be such as would give it stability in still water, so that then it would endeavour always to place (what we may term) its axis of equilibrium in a vertical position, then it follows that when the same particle floats on a wave, it will, in virtue of the same property of stability, endeavour to place its axis of equilibrium (not vertical, but) at right angles to the surface of the wave; so that if we were entitled to treat a ship as a mere surface particle in its relation to the wave, the position of momentary equilibrium would be thus completely defined—and this approximate view of the subject will be relied on in prosecuting the discussion.

Bearing in mind, indeed, how large are the dimensions of a thorough-bred Atlantic wave when compared with those of our largest three-decker, such a representation need not be deemed extravagant, the width of the ship not extending beyond 1-10th part of the wave space, and her draught not penetrating to a greater depth than the wave's height.

But in order to confirm our reliance on the results of this assumption, and to justify its application to smaller waves, it will be well to trace out, below the surface of the water, the operation of those conditions which have been shown to govern the dynamic relations of the surface itself; and this may be best done by comparing, as before, the action and reaction of the particles in undulating water, with their action and reaction when the water is at rest.

In still water, the same law in virtue of which the top surface is horizontal implies that there underlie the surface what may be described as level parallel strata, forming *plane* sub-surfaces of "uniform pressure." And exactly in the same way in undulating water, there must exist similar strata, forming *curvilinear* sub-surfaces of uniform pressure, the direction of these, at every point, being determined by the condition which has been shown to determine that of the upper surface of the wave, viz., that it must be everywhere at right angles to the resultant obtained by compounding the gravity of their moving particles with the accelerating forces which actuate them as deducible from their motions.

Now, it is easy to see that were the motions of the subjacent particles identical with those of the surface particles directly over them, the strata, or sub-surfaces of uniform pressure, would throughout be parallel to the upper surface of the wave, since in this case the resultant obtained by compounding the gravity of any subjacent particle with the accelerating force acting on it, as implied and measured by the accelerations it experiences, would be identical in direction with that deducible from the operation of the same conditions at the surface.

And the same result would follow, if the horizontal and vertical accelerations experienced by the subjacent particle, though not identical with those experienced by the surface particle, bore to them a certain due relation. For the direction of the resultant would be the same as at the surface, whenever it happened that the horizontal accelerating force acting on a subjacent particle (as indicated and measured by its horizontal accelerations) bore to its gravity + the vertical accelerating force acting on it (similarly indicated and measured), the same ratio which held good between the analogous conditions in the surface particle immediately above it.

It will not here be attempted to investigate the law which really governs the difference between the motions of the upper and the lower particles; and, indeed, it will be sufficient for the immediate purpose to view broadly those conditions which seem to determine the general character of the law.

I judge it to be dynamically impossible, in reference to any kind of wave, that in deep water the motion of the bottom particles can be even approximately the same as those of the surface particles; or that in water of unlimited depth the bottom particles can have any motion at all; or again, that in very shallow water, the motion of the particles at the bottom can fail to be considerable when compared with that of the particles at the surface. And though it is possible that the ratio may vary somewhat according to whether the wave be oscillating or translatory, I do not myself see any reason to expect such a variation.

Experimentally, indeed, it is patent that when oscillating waves are generated in a shallow channel, the particles at the bottom *appear* to move horizontally, as much as those at the surface; and the ripple mark which such waves create on a sandy bottom, may be summoned as a witness that the motion of the bottom particles is by no means inconsiderable; nor do I think that more than this can be said, when a wave of translation is generated in the same channel: and in fact it would be extremely difficult to arrange an experiment such as to justify a positive assertion that the motions of the top and bottom particles are absolutely identical. But though I do not admit their absolute identity, yet in a very shallow channel the motions are plainly so nearly equal, whether the waves be oscillatory or translatory, as to involve as a consequence the somewhat curious conclusion, that the virtual steepness of a wave in such a channel is greater in its lower regions than near the surface. So that a stabilised particle (a particle possessed of a definite axis of equilibrium which would become vertical in still water), must tend to assume a greater angle on the passage of a wave (or must have a more inclined position for its position of momentary equilibrium) when floating near the bottom than when floating at the surface. For the "virtual steepness" of the wave at any point in its interior is the slope of the corresponding stratum or sub-surface of uniform pressure, or (what is the same thing) is the inclination of the resultant which governs it. Now if we examine the motions of the particle, when floating near the bottom of the channel, we shall find that while its horizontal motions and their accelerations are, as has been already stated, very nearly as great as those which it would have possessed if floating near the surface, its vertical motion and their accelerations must be considerably less; indeed, if it were quite at the bottom, it would be incapable of vertical movement. It follows, therefore, that in a shallow channel, the ratio of the "horizontal accelerating force acting on the particle to the vertical "accelerating force acting on it + its gravity" will be greater and greater, and consequently the inclination of the resultant will be greater and greater, in proportion as the assumed position of the particle is nearer and nearer to the bottom.

Practically, however, the subject of the paper belongs to the region of deep water; and here, since the remoteness of the bottom prevents it from limiting the vertical motions of the particles, we may safely assume that the horizontal as well as the vertical motions are progressively less for subjacent than for surface particles. The rate of diminution cannot indeed be very rapid, since Atlantic storm-waves become sensibly modified in form on striking channel "soundings," so that plainly the particles of such waves possess considerable motion, even at the depth of 60 or 70 fathoms. And on the whole there appears no reason to assume that in waves of average proportions, the diminution follows such a law as will cause any sensible want of parallelism between the upper surface of the wave and the corresponding sub-surfaces of uniform pressure, at least within the depth which is reached by the displacement even of the largest ship.

As, then, it was shown to be rigorously true that to a stabilised particle, floating at any point on the upper surface of a wave, the position of momentary equilibrium is that which would place the axis of equilibrium of the particle at right angles to the tangent of the wave angle at the point where it floats, so it may be practically assumed that to another similar particle, floating or suspended at a moderate depth immediately below it, the position of momentary equilibrium is that which would place the axis of equilibrium of the lower particle parallel to that of the upper. And if we take account of the aggregation of particles for which a ship which displaces them is substituted, and of which she accepts the aggregate dynamic conditions, we know that her position of momentary equilibrium must be the mean of the positions belonging to the several particles displaced; and we may assume, with a close approximation to the truth, that this is the position which would place her axis of equilibrium (or we may say her mast) at right angles to the upper surface of the wave.

It follows, farther, that when the ship at any moment deviates from this position, the effort by which she endeavours to conform herself to it depends on the momentary angle of deviation, in the same manner as her effort to assume an upright position, when forcibly inclined in still water, depends on the angle of inclination. Hence her stability, *i.e.*, her effort to become vertical in still water, measures her effort to become normal to the waves in undulating water: and hence, just as when the ship floats in still water, this measure of the effort, changing with her changes of inclination, combined with the measure of her "moment of inertia," serves to determine her period of oscillation; so when she floats in waves, the effort, similarly measured, and changing not only with her own changes of inclination, but also with those of the travelling wave surface, serves to determine the successive changes of position which she will then experience.

It should be noticed that in adopting this view as exhibiting the fundamental law which governs the motion of a ship on waves, it becomes unnecessary to take separate account of the position of the ship's centre of gravity (*e.g.*, as whether it is above or below the waterline), or of the ship's peculiar form (as whether it "tumbles home," or spreads out "Symondite fashion" above the water); not because these conditions are irrelevant, but because in the exact degree in which they are relevant, they have been included, and full account has been taken of them in determining the ship's scale of stability in still water.*

In dealing with the problem of a ship oscillating in still water, it is usual to treat of small angles only, and to express the momentary force tending to place the ship in her position of rest by the equation $\phi = -\frac{W M \theta}{\rho}$ where (W) is the weight of the ship (say in tons); (M) the height of the metacentre above the centre of gravity (in feet); (ρ) the radius of gyration (in feet); (θ) the angle of inclination, and (ϕ) the force (in tons), which must be applied through the intervention of a "couple" whose span is = (ρ), in order to hold the ship at that angle. And if (ρ) be treated as an accelerating force, it follows

* I am aware that the proposition thus generally stated is markedly at variance with views which have been expressed by men whose authority deservedly ranks high on questions of Naval Architecture. It has been held, for example, by some, that though a plank, when laid flatwise on the water, would continue to accept the successive inclinations of surface brought under it by passing waves; the same plank, if loaded on its edge, so as to have a vertical position of rest in still water, would still float vertically when a wave passed under it; and again, that, carrying out the analogy to ships, while a flat and shallow vessel, deriving its stability from an extended plane of flotation, would endeavour to follow every undulation of the surface, a narrow deep one, on the contrary, deriving its stability from deeply stowed ballast, would endeavour to float always with its mast truly vertical. Whereas, according to the view here insisted on, the plank, whether floating flatwise or floating edgewise, the ship, whether stabilised by breadth of beam, or by deeply stowed ballast, would alike, at all times, endeavour to place its masts at right angles to the surface of the wave on which it rests.

I would more particularly direct the reader's attention to the degree and character of the difference between the views arrived at in this paper, and those expressed by Dr. Woolley in that able paper of his which naturally occupies so prominent a position in the first volume of the *Transactions* of this Institution; for it would be disrespectful towards one of such established reputation as a writer on this class of subjects, were I to allow the departure of these views from his, to be merged in a general admission of deviation from established beliefs. I will not, indeed, myself enter seriatim on the points of difference, for I could hardly do so without falling into somewhat of a controversial tone, but I will content myself with indicating, what I conceive to be the main characteristic of our respective modes of viewing the question. He seems to look to the effect of waves as striking a ship, and giving motion to her by the blow; while the basis of the view here brought forward is the idea of the ship as moving bodily *with* the wave, and virtually forming part of it; so that her oscillations are secondary or derivative results, governed by forces which the compound motion implies and expresses. I do not deny that, occasionally, and under peculiar circumstances, a wave may strike a ship and produce formidable effects. But I conceive that the ordinary and principal phenomena of rolling are practically independent of such a consideration.

that since $\frac{d^2\theta}{dt^2} = -\frac{g\phi}{W\rho}$ when we substitute for (ϕ) its value and reduce, we have $\frac{d^2\theta}{dt^2} = -\frac{gM}{\rho^2}\theta$, as the equation which expresses the rate at which the ship's angular velocity changes, in her effort to right herself when her momentary angle of inclination $= \theta$: and from its solution, we deduce (T), the time of a complete oscillation (say from starboard to port, or *vice versâ*), $T = \frac{\rho\pi}{\sqrt{gM}}$

Professor Moseley has shewn that experiment verifies this equation sufficiently, when the ship which is the subject of it is tolerably round-shaped, and is not encumbered with any unusual area of keel or deadwood; but that if she has a very sharp bottom or very deep keel, her actual period of oscillation is considerably greater than that indicated by the equation. And indeed if the mutual relations of the body of a ship and the contiguous masses of water be duly considered, it is sufficiently obvious that such a difference must arise in ships of such a form; because independently of the resistance (in the proper sense of that term) which the form must offer to the freedom of oscillation (in the same manner as with an ordinary pendulum, friction in the point of suspension, and resistance of air shorten the excursions without materially altering their periodic time), we see that the resisting areas must put in motion large masses of water which will continue to accompany them inertly, as if forming part of the body of the ship herself. And in order to frame an equation which would in such cases give a true result, it would be necessary to take account of these masses in determining the value of the moment of inertia, or the radius of gyration of the whole mass to be put in motion.

I cannot pretend to arrive directly at a measure of this condition, though the measure is at once furnished indirectly in any individual case, by comparing the observed with the calculated periodic time, and correcting the value of the radius of gyration accordingly. And if this comparison were repeated with a variety of typical forms, and the results were tabulated, some available scale of the relation would be established.

Again, it is usual to limit the scope of the equation within a small range of the angle (θ), because it is only within such a range that it can be applied with approximate truth to ships of very different forms. The applicability rests on the circumstance that for indefinitely small values of (θ) whatever be the form of the ship, the force tending to replace her in her position of rest, with her mast vertical, varies as (θ); which is, in fact, the essential condition of isochronous oscillation: and in fact, although it is easy to imagine a ship of such form that the actual law of force would deviate very widely from this, so soon as (θ) had begun to assume a tangible value, yet practically, shipbuilders adopt no such form, and so long as (θ) does not exceed 8° or 10°, the equation may be safely trusted as general for any ship we are likely to meet with.

In pursuing the investigation, however, it becomes necessary to contemplate much larger angles of excursion than these. But, on the one hand, the attempt to make the solution perfectly general, would add indefinitely to its difficulties, and to its complication; and, on the other hand, that form of ship which would be isochronous even to such angles of excursion, as 60° or 70°, not only may be accepted as a practically available form, but it is even typical of that which most approves itself to a practical eye as indicative of easy motion; it is, in fact, very nearly that of a three-decker, with the sides "tumbling "gracefully home."

I shall therefore assume in what follows, that I am dealing with a ship of such a form, since I shall thus both facilitate and shorten the inquiry; while the result arrived at though not strictly applicable, throughout, to ships of other forms, will indeed be strictly applicable to them for limited angles; and will at least indicate the character of the result which would follow when the angles are large: it will be found, too, that there is no great difficulty in subsequently forming an approximate conception of the nature and magnitude of the differences between the actual and the typical result under any circumstances.

It is not necessary to specify more precisely than has been already done the geometrical features of this typical form, because for the purposes of the investigation, the only property which need be taken account of, is that which secures the isochronism of oscillation, viz., that the force of stability tending to place the ship in the position of rest, *i.e.*, with her masts vertical, shall be directly as her deviation from that position (that is to say, directly as her angle of inclination), up to any angle which she will be supposed to attain in rolling; and on this hypothesis we may use, without limitation, the equations of motion for a ship oscillating in still water $\frac{d^2\,\theta}{dt^2} = -\frac{g\,\mathrm{M}}{\rho^2}\,\theta$, and $\mathrm{T} = \frac{\rho\,\pi}{\sqrt{g\mathrm{M}}}$, which were otherwise true for only limited values of (θ); and with equal freedom to apply the principles on which they rest to the case of a ship oscillating in undulating water, which is the ultimate object of the inquiry.

It will, however, facilitate the complete analysis of the motions of the ship in undulating water, if the nature of its oscillations in still water be more fully analysed and discussed, because it will be found that the expressions which define the latter, underlie, or are interwoven with those which define the former.

The laws to be thus investigated are, in fact, simply the laws of isochronous oscillation; since these are virtually the same, whether the excursions of the body to which they refer be rectilinear through a centre of force; or curvilinear towards or from a position of rest, as with a pendulum or the balance of a watch, or again with a suitably-formed floating body.

The law of force, which produces such oscillations, is, as has been said, that the force towards the position of rest varies as the distance of the body from that position, and it is pretty generally known that the intermediate parts of any given excursion are performed in times following the law of "sines;" that is to say, if a point be made to move uniformly in a circle, and to complete the circuit from one end of the diameter to the other while the body completes a single whole excursion, then at any intermediate moment the distance of the body from the position of rest will bear the same relation to its extreme distance as the sine of the angle which the revolving point has at the same moment reached bears to the radius of the circle. But this general law requires to be quantitatively expressed in terms of the various constants which enter into its structure, as these exist in our ship; and the solution shall be given in its most general form, though the process by which it is arrived at is thereby somewhat lengthened.

In applying, then, to the assumed isochronous ship the equation which expresses this law, we may say $\frac{d^2\theta}{dt^2} = -\frac{g\,M}{\rho^2}\theta$, without limitation as to the magnitude of (θ); and it will be remembered that in this equation, (θ) is the angle between the ship's axis of equilibrium and the vertical line; (g) the force of gravity $= \frac{32}{1''^2}$; M the height of the ship's metacentre above her centre of gravity; and (ρ) the radius of gyration, so measured as to take due account of the masses of water whose motion is involved in hers.

For convenience of solution, the equation may be written, (1) $\frac{d^2\theta}{dt^2} = -n^2\theta$.

Then, $\frac{d\theta\, d^2\theta}{dt^2} = -n^2\theta\, d\theta$, and integrating $\frac{1}{2}\frac{d\theta^2}{dt^2} = -\frac{n^2}{2}\theta^2 + c$;

Or, as we may write for convenience, (2) $\frac{d\theta^2}{dt^2} = -n^2\theta^2 + c^2$;

$$\text{Hence, } \frac{d\theta}{dt} = \pm\sqrt{c^2 - n^2\theta^2} = \pm\, n\frac{c}{n}\sqrt{1 - \frac{n^2}{c^2}\theta^2};$$

$$\text{Or, } \frac{\frac{n}{c}d\theta}{\pm\sqrt{1-\frac{n^2}{c^2}\theta^2}} = n\,dt.$$

$$(3) \quad \text{and} \int \frac{\frac{n}{c}\,d\theta}{\pm\sqrt{1-\frac{n^2}{c_2}\theta^2}} = n\,t + c'.$$

To integrate the expression under the integral sign let $\sin^{-1}\left(\frac{n}{c}\theta\right) = z$; or, $\frac{n}{c}\theta = \sin z$.

$$\text{Then } \frac{n}{c}\,d\theta = \cos z\,dz = \pm\sqrt{1-\sin^2 z}\;dz = \pm\sqrt{1-\frac{n^2}{c^2}\theta^2}\;dz.$$

$$\text{So that } \frac{\frac{n}{c}\,d\theta}{\pm\sqrt{1-\frac{n^2}{c^2}\theta^2}} = dz, \text{ and} \int \frac{\frac{n}{c}\,d\theta}{\pm\sqrt{1-\frac{n^2}{c^2}\theta^2}} = z = \sin^{-1}\frac{n}{c}\theta.$$

Hence, recurring to (3), and substituting, $\sin^{-1}\frac{n}{c}\theta = n\,t + c'$; or, $\frac{n}{c}\theta = \sin(n\,t + c')$. (4.)

Here, since (θ) becomes impossible when $n\ (\theta)$ becomes larger than (c), its alternately positive and negative values are those of continued oscillations, whose amplitude is limited by that condition.

It follows, also, from (4) that any given positive value of (θ) is repeated whenever $(n\ t + c')$ is of the form $(4\ m + 1)\ \frac{\pi}{2} \pm \gamma$, (m) being any positive integer and (γ) some angle less than $\frac{\pi}{2}$. And the same value will be repeated, but with a negative sign, when $(n\ t + c')$ is of the form $(4\ m + 3)\ \frac{\pi}{2} \pm \gamma$.

Now, in order that a given phase of oscillation may be repeated in its alternate negative and positive aspect, and that the intervening time may be determined, it is plain that $(n\ t + c')$ must step either from $(4\ m + 1)\ \frac{\pi}{2} + \gamma$ to $(4\ m + 3)\ \frac{\pi}{2} + \gamma$, or, again, from the latter to $\left\{ 4\ (m + 1) + 1 \right\} \frac{\pi}{2} + \gamma$; or *vice versâ*, from $(4\,m + 1)\ \frac{\pi}{2} - \gamma$ to $(4\ m + 3)\,\frac{\pi}{2} - \gamma$, or, again, from the latter to $\left\{ 4\ (m + 1) + 1 \right\} \frac{\pi}{2} - \gamma$. And if t_1 and t_2 be the two consecutive values of (t) corresponding with the transition, then $t_2 - t_1$ will be the interval: and we have $(n\,t_2 + c') - (nt_1 + c') = \left\{\overline{4\ m + 3} - \overline{4\ m + 1}\right\} \frac{\pi}{2}$, or, secondly $= \left\{\overline{4\ (m + 1) + 1} - (4\ m + 3)\right\} \frac{\pi}{2}$, and in either case, $n\,(t_2 - t_1) = \frac{2\,\pi}{2}$; or if we call $(t_2 - t_1)$, or the period, $=$ T, then $\mathrm{T} = \frac{\pi}{n}$. That is to say, the alternating phase of oscillation will be repeated in the definite period of $\frac{\pi}{n}$, however the phase be selected, and whether the maximum amplitude of the oscillation be large or small. It may be observed, in passing, that we may at any time substitute for (n) its value $= \frac{\pi}{\mathrm{T}}$ derived from this condition; and, also, that since in equation (1), n^2 stands for $\frac{g\,\mathrm{M}}{\rho^2}$, it follows that $\mathrm{T} = \frac{\pi\,\rho}{\sqrt{g\,\mathrm{M}}}$.

To determine the constants (c) and (c'), differentiate equation (4) and we have $\frac{n}{c}\ d\theta =$ $\cos\,(n\ t + c')\ d\ (n\,t + c')$, or $\frac{d\theta}{dt} = c\cos\,(n\,t + c')$. Now let $t = 0$ give $\frac{d\theta}{dt} = \mathrm{U}$ (that is to say, let the ship be moving with a given angular velocity when we begin to take account of the oscillation);

$$\therefore \quad \mathrm{U} = c\,\cos\,c' \qquad (5).$$

Again, let $t = 0$ give $\theta = \alpha$ (that is to say let the ship have a definite angle of inclination when we begin to take account of the oscillation);

$$\therefore \quad \text{by (4)}\ \alpha = \frac{c}{n}\,\sin\,c' \qquad (6).$$

Expanding (4), $\theta = \frac{c}{n}$ $(\sin nt \cos c' + \cos nt \sin c')$, and substituting for $(c \cos c')$ and for $(\frac{c}{n} \sin c')$ their values from (5) and (6), and putting for n its value $= \frac{\pi}{\mathrm{T}}$, we arrive at the equation

$$\theta = \frac{\mathrm{UT}}{\pi} \sin \frac{\pi t}{\mathrm{T}} + \alpha \cos \frac{\pi t}{\mathrm{T}} \qquad (7.)$$

which is the complete solution of the equation $\frac{d^2 \theta}{dt^2} = -n^2 \theta$; in which equation we may also substitute for n its value $\frac{\pi}{\mathrm{T}}$, so that it becomes $\frac{d^2 \theta}{dt^2} = -\frac{\pi^2}{\mathrm{T}^2} \theta$.

It should be added, that if in dealing with equation (7), (the complete solution), we choose to date our account of time from a moment at which the ship is vertical, so that $\alpha = 0$, the expression becomes $\theta = \frac{\mathrm{UT}}{\pi} \sin \frac{\pi t}{\mathrm{T}}$, which shows that when the maximum angle, or extreme point of the oscillation is reached (*i.e* when $\frac{t}{\mathrm{T}} = \frac{1}{2}$, and $\sin \frac{\pi t}{\mathrm{T}} = 1$) the angle of inclination, say Θ, $= \frac{\mathrm{T}}{\pi} \times$ the maximum angular velocity; and that at any other time (t), the angle of inclination follows what has already been described as the "law of sines."

We see, then, that though the effort of stability of any given ship depends primarily on her mass, and the position of her metacentre and her centre of gravity, the rate at which she will acquire or lose velocity under given circumstances of inclination and angular velocity, and the position she will assume at any period, may be wholly expressed in terms of her "periodic time" T.

In making this statement, I do not forget that the oscillations of a ship are performed in a resisting medium, and that the scale of resistance is so high that, when a ship of whatever form has been set rolling in still water, the range of each successive oscillation becomes sensibly less than that of its predecessor. But the remarks which this condition requires will, for the most part, answer their purpose better if reserved to a later period of the inquiry.

In proceeding from the case of the ship oscillating in still water to that of the ship oscillating in undulating water, it is necessary to remind the reader, that it has been shown that the momentary effort of the ship is to place her masts at right angles to the surface of the wave where she floats; and that for a given ship occupying, at any moment, an angle of inclination differing from this, the measure of the effort is the same as that by which she would endeavour to assume a vertical position if occupying for the moment, in still water, an inclined one, with an angle equal to that difference. Hence, if θ' be the inclination of the wave surface, *i.e.* its deviation from the horizontal plane, just as the effort of stability in still water was resolved into the expression $\frac{d^2 \theta}{dt^2} = -\frac{\pi^2}{\mathrm{T}^2} \theta$, so it follows that in undulating water it is expressed by the equation $\frac{d^2\theta}{dt^2} = -\frac{\pi^2}{T^2}(\theta - \theta')$; which will assume a form suitable for integration if (θ') be eliminated in terms of (t) so as to express the changes of inclination of the wave surface; and the elimination could be performed rigorously, if the true equation of an oscillating wave were certainly known.

I am not, indeed, aware whether this question has yet been thoroughly solved; I believe, however, we may safely take the equation of the "curve of sines" as approximately representing the shape of large and uniformly-recurring ocean waves. Then, if we suppose a series of such waves to travel past the ship with given uniform velocity, the value of (θ') for any given point in the series, which in its turn arrives at the ship, will be immediately determinable in terms of the corresponding value of (t), because the horizontal distance along the curve from its assumed starting point to the point in question, while it determines (θ') in virtue of the differential coefficients of the curve, is itself measured by (t); since, when the velocity of transit is given, the time occupied in the performance of a given distance at once follows.

I am well aware that there are good reasons for believing that in respect of mere profile, the wave would be more accurately represented by some member of the cycloidal or trochoidal family, than by the curve of sines, their near relation; but were we to adopt the trochoidal hypothesis, it would be necessary to take account of certain conditions of action, the introduction of which certainly tends to bring its ultimate result into much closer correspondence with that deducible directly from the curve of sines, than would at first sight be expected on observing the points of difference between the figures of the two curves. I have not yet been able to master the mathematical difficulties which this mode of treatment involves, though I am not without hopes of succeeding in the attempt; I think, too, I see my way to a method of applying a rigorous experimental test to the results which may be arrived at; but, for the present, I content myself by describing the character of the differences between the two curves, and showing their tendency; and I hope that on the strength of this, the reader will be content to accept the curve of sines as the basis of a sufficiently approximate solution.

Viewing their features generally, the characteristic difference between the two classes of curve is, that while with the curve of sines, the curvature of the hollow of the wave downward, is represented as identical with that of the crest upward; with any one of the trochoidal family the upward curvature of the apex is, in a greater or less degree, more sharp than the downwards curvature of the hollow, the degree varying with the proportion of length of wave to height of wave, and ranging from the case of an extremely long low trochoidal wave, in which the figure is scarcely distinguishable from that of the curve of sines, up to that of the pure cycloidal wave, the length of which, from crest to crest, is $(\pi) \times$ its total height, and in which the apex forms a perfectly developed cusp or angle, representing the conditions of an almost imminent breaker. Hence, it would at first sight seem that if the trochoidal hypothesis be true, a ship must experience more abrupt changes of wave effort near the crest, than near the hollow; and that the use of the curve of sines hypothesis must lead to error, in consequence of its failing to take account of any such difference. But this appearance is in a great measure delusive, owing to the difference between (what I venture to term) the "*hydrostatic tension*" of the water at the top, and bottom of the wave. And as this is a condition, the nature of which has not, as far as I am aware, been referred to by others who have treated of the question to which it belongs, I am the more anxious to direct attention to it.

Without attempting to go into refinements, it is indisputable that, on the whole, the particles which form alternately the hollow and the crest of the wave, alternately rise and fall during the change; and it will not be questioned that the points of maximum velocity in each direction, exist somewhere in the intermediate height of the wave. It follows, that in the lower parts of the wave, the particles are being retarded in their downward velocity, or are accumulating upward velocity; and either of these conditions implies that during their maintenance, the particles are being pushed in an upward direction, and are consequently pushing against each other with more than their mean force; that, for instance, whereas at the depth of a foot below the surface in still water, the particles press against each other or against any immersed body with a force of ·43 lbs. per square inch, at a similar depth below the hollow of a wave the mutual pressure would be more than ·43 per square inch; and as in the upper portion of the wave the converse condition must hold, and the particles are either losing upward velocity, or are being accelerated downwards, so that part of the effect of the natural gravity is thus absorbed, it follows that their mutual pressure is less than its mean or natural amount, and at the depth of 1 foot below that surface, it would fall short of ·43 lbs. per square inch.

This will be perhaps best understood by considering what must happen if a bucket of water be attached to the top of the piston rod of a vertically-acting steam engine, a position which, so far as vertical motion is concerned, would represent fairly the circumstances of a similar volume of water when undergoing wave motion; in such a case, by varying at pleasure the speed of the reciprocations, we might easily arrive at a velocity such that, at the summit of each stroke, the bucket should be actually drawn away from its contents faster than they could follow it in virtue of gravitation; or as an example of a more quantitative character, we might select exactly such a speed of reciprocation that the contents of the bucket would just, and only just, keep company with it at the summit of the stroke. On this supposition it is plain that the particles of water would at the moment absolutely fail to press at all, either against the bottom or sides of the bucket, or against each other, or finally against any immersed floating body; though at the same time, such a body would not acquire any increased immersion for want of support, since itself would, *pari ratione*, be divested of its power of pressing against the particles of water, exactly in the same way as these had, by the same cause, been incapacitated from pressing against each other and against it.

But though the loss of support would thus not cause any increase of immersion, it would cause an absolute loss of stability in the floating body, however stable it might be in stationary water. A model boat floating in the water, when passing through this phase of motion, might be blown over by the very slightest puff of wind in her sails.

Similarly at the end of the down and the commencement of the up-stroke, if the same speed of reciprocation were maintained, the mutual pressure of the particles of water against each other, and their pressure against the sides and bottom of the bucket, or against any floating body, would be precisely doubled, in virtue of the same conditions which, it was shown, would precisely neutralise their pressure at the end of the up and the commencement of the down stroke; and the stability of the floating body, which there consequently vanished, would here be doubled. The model boat would stand up under twice as great a pressure of wind as she would bear if in still water.

Now the speed of reciprocation which has here been assigned to the vertically moving volume of water, corresponds exactly with that of the vertical component of the motion in the particles forming respectively the crest and hollow of a purely cycloidal wave; and the absence of mutual pressure in the particles at the crest of the wave corresponds with the circumstance that such a wave is on the verge of breaking; so that a ship floating on such waves would have her stability doubled when in the trough, and if her dimensions were very small compared with those of the wave, it is not a paradox to say her stability would absolutely vanish as she floated over the crest.

Short of this (the extreme result, due to the greatest possible speed of vertical reciprocations which wave motion can exhibit), we must expect in waves of more or less abrupt form, the stability of a floating body to be alternately diminished and increased on the same principle, and in a degree proportioned to the speed of the reciprocation. And since it has been shown that a ship's stability measures the force by which when, with her mast at a given inclination to the wave surface, she endeavours to place it at right angles to that surface, it follows that, on the whole, the less steep parts of the trochoidal wave (being in the trough or hollow) are more effective in giving motion to the ship; steeper parts (being near the crest) are less effective than the degree of steepness in each case would, at first sight, lead us to expect; and this difference corresponds, in some degree, with the result which we obtain by substituting the equation of the curve of lines for the equation of the trochoid, and at the same time discarding from the question the changes of hydrostatic tension: for thus, instead of the less steep, but more effective slopes of the trochoidal hollow, we take simply the somewhat steeper slope of the hollow of the curve of sines; and instead of the steeper, but less effective slopes of the trochoidal wave crest, we take simply the somewhat flatter crest exhibited by the curve of sines.

One other instance must be mentioned, in which the use of the curve of sines, as the wave equation, at once supplies the equivalent of a condition which it would be necessary to take separate account of in proceeding on the trochoidal hypothesis.

This hypothesis correctly represents the fact, that the surface wave particles, or any floating body substituted for a small aggregation of particles, oscillate backward and forward as well as upward and downward; each describing an exact and complete circle in the course of each complete wave recurrence. It follows that in determining the "time" due to any portion of the wave curve as it passes the floating body (with the object of expressing (θ') or the slope of the wave, in terms of (t)), we must take account of the horizontal displacement of the floating body (let us say the ship) which meanwhile occurs, and correct the "time account" accordingly, by deducting or adding the result (as the case may require), to the time which would be *primâ facie* required, for the transit of each corresponding geometrical interval between the two points on the wave, which the ship occupied at the beginning and end of the account.

Thus, if we commence by supposing the ship to be on the middle of the wave on the descending side, then, according to the trochoidal hypothesis, she has, at the moment, a simply vertical motion, downwards; but as the hollow of the wave approaches her, her circular orbit carries her bodily towards it, and when she is at the bottom of the hollow, she will have moved through a quarter of the circle, and the horizontal component of her motion will have carried her through half the diameter of the circle to meet the wave. Again, when she is at the middle of the wave height on the ascending side, she will have completed a second quarter of the circle, and will have travelled backward through the remainder of the diameter to meet the wave. In arriving at each of these points, therefore, she will have occupied a time proportionably less than that occupied by the transit of the corresponding portion of wave curve past a stationary point, the difference being, in each case, the time due to the horizontal distance travelled by the ship, valued according to the wave's velocity. In the same way, she will now have a motion simply vertical upwards; but as the crest of the wave approaches her, she will begin to recede bodily from it: when she reaches the summit, she will have progressed through a quarter of a circle, or by half its diameter, towards her original position, and when half the wave-back has passed she will have resumed that position precisely. And it is obvious that in resuming it, she will have occupied a time proportionably greater than that due to the transit of the corresponding portion of wave curve past a stationary point; lengthening in fact her time of passing the wave summit, as much as she had shortened that of passing the wave hollow.

Now it is obvious at once, that this "time correction" corresponds, in a general way, with the result obtained by the use of the curve of sines equation. For if a curve of sines wave, and a trochoidal wave, of equal height, length, and velocity, be supposed to move simultaneously past a fixed point, it is plain that the hollow of the former, being the shorter, will be traversed in the shorter time; its crest, being the longer, will be traversed in the longer time.

But on closer examination, it appears that the correction thus indirectly supplied is not merely of the required character, but is precisely of the required amount, and that if we assume one ship to be rising and falling vertically on the assumed curved sines wave, and another to be following her true circular path on the corresponding trochoidal wave, supposing them to start simultaneously at the same level, each will continue to attain the same level at the same instant of time throughout the whole series of wave phases. This statement is one purely geometrical, and it follows at once from a comparison of the modes in which the two curves are respectively generated. For in both curves the vertical ordinates are derived for the motion of a point, supposed to move with uniform circumferential velocity in a circle whose diameter is the height of the wave from hollow to crest; while the centre of the circle is supposed to move horizontally with uniform velocity, and to complete a space equal to the length of the wave from crest to crest when the point completes the whole circumference. In the curve of sines, the horizontal length of the portion of curve thus generated in a given time is simply the corresponding horizontal travel of the centre of the generating circle; while in the trochoid, the horizontal length of the portion of curve generated in a given time is the horizontal travel of the centre + the linear sine of the angle, travelled over in the arc of the generating circle by the generating point. But in both curves alike the vertical ordinate due to this length of curve (measuring it from the path of the travelling centre) is the linear cosine of that arc. So that counting position by time, the assumed method of treating the wave as a curve of sines with the ship rising and falling vertically, assigns to her at any moment the same level on the wave surface which would be assigned to her by treating the wave rigorously on the trochoidal hypothesis.

It follows also, from the comparative nature of the two curves, that, at the middle height of the wave, the wave angle is identically the same on either hypothesis: it is below this level, that the trochoidal wave has, as before pointed out, the flatter angles; it is above this level that it has the steeper angles; differences which, as was also shown, the difference of "hydrostatic tension" between hollow and crest tended to compensate, tending also thereby to justify the use of the curve of sines hypothesis, which, with its steeper hollow and flatter summit, has provided a compensation of a somewhat similar character.

It may be perhaps said that it is childish to plead "tendencies" in reference to questions which ought to receive quantitative solutions; and this would be admitted if the quantitative solution were attainable, or if any great stress were laid on the alleged tendency. Unfortunately, for my own part, I cannot at present do more than hope for the complete solution; and while it seemed necessary to point out the real defects of the hypothesis adopted, (though it is felt that they are not of very serious moment,) it seemed also that something was gained by pointing out that the defects tended virtually to modify rather than intensify each other.

I, therefore, proceed to the direct solution of the question, on the curve of sines hypothesis; recurring to the equation of motion given in p. 193, $\frac{d^2\theta}{dt^2} = -\frac{\pi^2}{T^2}(\theta - \theta')$, in reference to which it may save the reader trouble if I remind him that (θ) is the angle between the ship's mast and the vertical line, (θ') the slope of the wave or its deviation from the horizontal, and T the time in which the ship completes a single independent oscillation in still water, say from starboard to port, or vice versâ.

To adapt the equation of the curve of sines to the purpose in view, let (H), or the height of the wave from hollow to crest, be the diameter of the generating circle, placed vertically; let (L) be the length of the wave from hollow to crest; and let (T′) be the "period" of the wave, or the time occupied by it in traversing the space (L): in this period the generating point will have traversed the arc of the vertically placed semicircle. Then, at any intermediate period (t), the generating point will have traversed an arc (θ), the value of which is $\alpha = \frac{\pi t}{T'}$: then, calling the space traversed by the wave and the height to which it has risen in time (t), (l), and (h), respectively, we have the following equations:—

$$(a) \qquad l = L\frac{t}{T'}, \text{ and } \frac{dl}{dt} = \frac{L}{T'}.$$

$$(b) \qquad h = \frac{H}{2}\left(1 - \cos\frac{\pi t}{T'}\right), \text{ and} \frac{dh}{dt} = \frac{H}{2}\,\frac{\pi}{T'}, \sin\frac{\pi t}{T'}.$$

$$(c) \quad \text{So that, } \frac{dh}{dl} = \frac{\pi}{2}\,\frac{H}{L}\sin\frac{\pi t}{T'}.$$

Now, $\frac{dh}{dl} = \tan\theta'$, ($\theta'$) being the slope of the wave; and since, generally, this is not large, and since it is also a fact that when the wave slope is very steep, it must be under circumstances which create, in a high degree, that loss of hydrostatic tension which has already been explained, no large error can be introduced by treating θ' as practically $= \tan\theta'$; an assumption which while it has the negative merit of not being very wide of the truth, has also the positive merit of rendering the equation of wave motion capable of integration. It should be noted in passing, that if $\frac{dh}{dl}$ be taken as the slope of the wave, it turns out that the maximum value of this occurs when (t) $= \frac{T'}{2}$; then $\sin\frac{\pi t}{T'} = \sin\frac{\pi}{2} = 1$; hence $\frac{\pi}{2}\,\frac{H}{L}$ is the steepest slope of the wave, and this is placed at the middle height, and the middle distance, of the ascending, or descending side.

In the equation of motion, then, which determines the changes of the ship's position, we may substitute for (θ') its value, $\frac{\pi}{2}\,\frac{H}{L}\sin\frac{\pi t}{T'}$, and the equation becomes

$$\frac{d^2\theta}{dt^2} = -\frac{\pi^2}{T^2}\left(\theta - \frac{\pi}{2}\,\frac{H}{L}\sin\frac{\pi t}{T'}\right) : (1.)$$

and the assumed conditions of commencement, with reference to which the integration is to be completed, are, that at the bottom of some given wave, which is followed by a series of waves having the same shape and period; the ship, having her broadside to the waves, is inclined at some given angle, and is rolling with some given angular velocity; which angle and which velocity will be introduced in due time into the solution, as the constants appropriate to the corresponding steps in the integration.

For convenience the equation may be written,*

$$\frac{d^2\theta}{dt^2} + n^2\theta - B\sin kt = 0. \quad (2.)$$

Or, as more suitable for integration, multiplying by dt,

$$d\left(\frac{d\theta}{dt}\right) + n^2\theta\, dt - B\sin kt\, dt = 0.$$

Then, in order to facilitate integration, multiplying by $\cos nt$, and taking the integral form, we have,

$$\int\cos nt\, d\left(\frac{d\theta}{dt}\right) + n^2\int\cos nt\,\theta\, dt - B\int\cos nt\sin kt\, dt = 0. \quad (3.)$$

Integrating the first term by parts, and proceeding with integration by parts,

$$\int\cos nt\, d\,\frac{d\theta}{dt} = \cos nt\,\frac{d\theta}{dt} - \int -\sin nt\; n\, dt\,\frac{d\theta}{dt}$$

$$= \cos nt\,\frac{d\theta}{dt} + n\sin nt\,\theta - \int n^2\cos nt\,\theta\, dt. \quad (4.)$$

Proceeding in the same manner with the third term of equation (3)

$$\int\cos nt\sin kt\, dt = \frac{1}{n}\sin kt\sin nt - \int\frac{k}{n}\sin nt\cos kt\, dt$$

$$= \frac{1}{n}\sin kt\sin nt + \frac{k}{n^2}\cos kt\cos nt + \frac{k^2}{n^2}\int\cos nt\sin kt\, dt;$$

hence,

$$\int\cos nt\sin kt\, dt\left(1 - \frac{k^2}{n^2}\right) = \frac{1}{n}\sin kt\sin nt + \frac{k}{n^2}\cos kt\cos nt$$

and $$\int\cos nt\sin kt\, dt = \frac{n^2}{n^2-k^2}\left\{\frac{1}{n}\sin kt\sin nt + \frac{k}{n^2}\cos kt\cos nt\right\}. \quad (5.)$$

Substituting from (4) and (5) in (3), the middle term of the latter equation is obliterated, and we have,

$$\cos nt\,\frac{d\theta}{dt} + n\sin nt\,\theta - B\,\frac{n^2}{n^2-k^2}\left\{\frac{1}{n}\sin kt\sin nt + \frac{k}{n^2}\cos kt\cos nt\right\} + C = 0. \quad (6.)$$

To integrate this, multiply by dt and divide by $\cos^2 nt$,

then, $$\frac{\cos nt\, d\theta + n\sin nt\,\theta\, dt}{\cos^2 nt} - \frac{B}{n^2-k^2}\,\frac{n\sin kt\sin nt + k\cos nt\cos kt}{\cos^2 nt}\, dt - \frac{C}{\cos^2 nt}\, dt = 0;$$

* To my friend, Mr. Bell, whose assistance in this inquiry I have already had occasion to acknowledge heartily, I am indebted for the following integration, the key to which he lighted on in a volume of Professor Airy's. I have slightly altered it from the form in which I received it, to one which, in some of its steps, appeared to myself somewhat clearer; I only hope he will not consider that I have attempted to secure it as my own by spoiling it.

and taking the integral, (the two first terms are complete differentials),

$$\frac{\theta}{\cos n\,t} - \frac{\mathrm{B}}{n^2-k^2}\,\frac{\sin k\,t}{\cos n\,t} + \mathrm{C}\tan n\,t + \mathrm{C}' = 0;$$

Hence the solution is,

$$\theta = \frac{\mathrm{B}}{n^2-k^2}\sin k\,t + \mathrm{C}\sin n\,t + \mathrm{C}'\cos n\,t. \quad (7.)$$

To determine C,

$$\frac{d\,\theta}{d\,t} = \frac{\mathrm{B}\,k}{n^2-k^2}\cos k\,t + \mathrm{C}\,n\cos n\,t - \mathrm{C}'\,n\sin n\,t,$$

then let

$$\frac{d\,\theta}{d\,t} = \mathrm{U} \text{ when } t = 0 \;\therefore\; \mathrm{U} = \frac{\mathrm{B}\,k}{n^2-k^2} + \mathrm{C}\,n;$$

$$\text{or,}\quad \mathrm{C} = \frac{v}{n} - \frac{k}{n}\,\frac{\mathrm{B}}{n^2-k^2}.$$

Again, to determine C′,

$$\text{let } \theta = a, \text{ when } t = 0.$$

Then by (7)

$$a = \mathrm{C}'.$$

Substituting these values for C and C′, we have

$$\theta = \frac{\mathrm{B}}{n^2-k^2}\sin k\,t + \left\{\frac{\mathrm{U}}{n} - \frac{k}{n}\,\frac{\mathrm{B}}{(n^2-k^2)}\right\}\sin n\,t + a\cos n\,t. \quad (8.)$$

Which is the complete solution of equation (2).

If, now, we compare equation (2) with equation (1), we have

$$n = \frac{\pi}{\mathrm{T}},\ \mathrm{B} = \frac{\pi}{2}\,\frac{\mathrm{H}}{\mathrm{L}}\,\frac{\pi^2}{\mathrm{T}^2} \text{ and } k = \frac{\pi}{\mathrm{T}'} \text{ and} \therefore \frac{\mathrm{B}}{n^2-k^2} = \frac{\pi}{2}\,\frac{\mathrm{H}}{\mathrm{L}}\,\frac{1}{1-\frac{\mathrm{T}^2}{\mathrm{T}'^2}} \text{ and } \frac{k}{n} = \frac{\mathrm{T}}{\mathrm{T}'}.$$

And if these values are replaced in (8), the solution becomes

$$\theta = \frac{\pi}{2}\,\frac{\mathrm{H}}{\mathrm{L}}\,\frac{1}{1-\frac{\mathrm{T}^2}{\mathrm{T}'^2}}\sin\frac{\pi\,t}{\mathrm{T}'} + \left\{\frac{\mathrm{U\,T}}{\pi} - \frac{\frac{\pi}{2}\,\frac{\mathrm{H}}{\mathrm{L}}\,\frac{\mathrm{T}}{\mathrm{T}'}}{1-\frac{\mathrm{T}^2}{\mathrm{T}'^2}}\right\}\sin\frac{\pi\,t}{\mathrm{T}} + a\cos\frac{\pi\,t}{\mathrm{T}}$$

$$= \frac{\pi}{2}\,\frac{\mathrm{H}}{\mathrm{L}}\,\frac{1}{1-\frac{\mathrm{T}^2}{\mathrm{T}'^2}}\left(\sin\frac{\pi\,t}{\mathrm{T}'} - \frac{\mathrm{T}}{\mathrm{T}'}\sin\frac{\pi\,t}{\mathrm{T}}\right) + \frac{\mathrm{U\,T}}{\pi}\sin\frac{\pi\,t}{\mathrm{T}} + a\cos\frac{\pi\,t}{\mathrm{T}}. \quad (9.)$$

Now, in discussing the simple oscillations of a ship in still water, in the earlier part of the paper we found that if U be her angular velocity, and (a) her angle of position, when $t = 0$

$$\theta = \frac{\mathrm{U\,T}}{\pi}\sin\frac{\pi\,t}{\mathrm{T}} + a\cos\frac{\pi\,t}{\mathrm{T}}.$$

And if this expression be compared with the two final terms of the equation (9), which expresses the value of (θ) for a ship rolling in waves, it will be seen to be identical with the two final terms of the latter. And, on the other hand, if the constants U and (a) vanish, that is to say, if we assume that the ship was stationary and upright when the waves reached her, she will undergo a series of movements defined by the expression

$$\theta = \frac{\pi}{2} \frac{\mathrm{H}}{\mathrm{L}} \frac{1}{1 - \frac{\mathrm{T}^2}{\mathrm{T}'^2}} \left(\sin \frac{\pi t}{\mathrm{T}'} - \frac{\mathrm{T}}{\mathrm{T}'} \sin \frac{\pi t}{\mathrm{T}} \right); \quad (10.)$$

which series, though its results have to be combined with those of the series expressing the ship's proper oscillations due to a previously existing velocity and position, when such are assumed to have existed, maintains nevertheless its independent vitality and integrity; each series, in fact, thus retaining its complete individuality, in a manner analogous to what may be observed to happen when independent sets of wave oscillations in the water surface intersect or overtake each other.

The simplest and clearest method, therefore, of tracing out the combined result is, for the most part, to trace the results of each series separately, and observe how these mutually modify each other without any interference in their separate sources: and, in fact, we may conceive the ship to perform this operation for herself—at each instant she occupies exactly the position in which the waves would have placed her, except that she has also made meanwhile exactly the motion which she would have made had she continued to move in still water.

A full analysis of the results which issue from the combination of these two perfectly independent conditions would of course run into interminable variations. It will, however, enable us to appreciate the general character of such results if we examine those derived from certain critical phases of the conditions.

(1.) Let us suppose the ship to have been stationary and upright when the first wave reached her, so that $\mathrm{U} = 0$ and $\alpha = 0$, when, as already pointed out, the equation becomes

$$\theta = \frac{\pi}{2} \frac{\mathrm{H}}{\mathrm{L}} \frac{1}{1 - \frac{\mathrm{T}^2}{\mathrm{T}'^2}} \left\{ \sin \frac{\pi t}{\mathrm{T}'} - \frac{\mathrm{T}}{\mathrm{T}'} \sin \frac{\pi t}{\mathrm{T}} \right\}. \quad (10.)$$

And let us farther assume that the period of the ship's natural roll and the period of the wave are the same, so that $\mathrm{T} = \mathrm{T}'$. On these suppositions the equation assumes the form $\theta = \frac{0}{0}$, and we have to deduce the proper limiting value of this expression.

We may do this by assumiug $\mathrm{T} = \mathrm{T}' + h$, h being as small as we please.

$$\text{Then} \quad \theta = \frac{\pi}{2} \frac{\mathrm{H}}{\mathrm{L}} \frac{1}{1 - \frac{\mathrm{T}'^2 + 2\mathrm{T}'h + h^2}{\mathrm{T}'^2}} \left(\sin \frac{\pi t}{\mathrm{T}'} - \frac{\mathrm{T}' + h}{\mathrm{T}'} \sin \frac{\pi t}{\mathrm{T}' + h} \right).$$

Now, if h be very small,

$$1 - \frac{\mathrm{T}'^2 + 2\mathrm{T}'h + h^2}{\mathrm{T}'^2} = 1 - 1 - \frac{2h}{\mathrm{T}'}, \text{ also } \frac{\pi t}{\mathrm{T}' + h} = \frac{\pi t}{\mathrm{T}'} - \frac{\pi t}{\mathrm{T}'^2} h,$$

and we have

$$\cos \frac{\pi t}{T'^2} h = 1, \text{ and } \sin \frac{\pi t}{T'^2} h = \frac{\pi t}{T'^2} h;$$

the equation, therefore, becomes

$$\theta = \frac{\pi}{2} \frac{H}{L} \frac{1}{1-1-\frac{2h}{T'}} \left\{ \sin \frac{\pi t}{T'} - \sin \frac{\pi t}{T'} + \frac{\pi t h}{T'^2} \cos \frac{\pi t}{T'} - \frac{h}{T'} \sin \frac{\pi t}{T'} + \frac{\pi t h^2}{T'^3} \cos \frac{\pi t}{T'} \right\}$$

$$= \frac{\pi}{2} \frac{H}{L} \frac{T'}{-2h} \left\{ \frac{\pi t h}{T'^2} \cos \frac{\pi t}{T'} - \frac{h}{T'} \sin \frac{\pi t}{T'} \right\}$$

$$= \frac{\pi}{4} \frac{H}{L} \left\{ \sin \frac{\pi t}{T'} - \frac{\pi t}{T'} \cos \frac{\pi t}{T'} \right\}. \quad (11.)$$

It will be remembered that $\frac{\pi}{2} \frac{H}{L}$ is the maximum slope of the wave, that is, its slope at its middle height, which we may call Θ'.

Bearing this in mind, we observe in equation (11) that all the phases of the oscillations which it represents must in form recur with the recurring phases of each successive wave.

So often as $\cos \frac{\pi t}{T'} = 0$, $\left(\text{that is to say, when } \frac{\pi t}{T'} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \&c., \right)$ $\sin \frac{\pi t}{T'}$ is alternately $\pm$ 1, and the equation becomes

$$\theta = \pm \frac{1}{2} \Theta'.$$

That is to say, at the middle height of the wave the ship's masts will have an inclination $= \frac{1}{2}$ the slope of the wave at this point, which is also the maximum slope.

But what is most important is that when $\sin \frac{\pi t}{T'} = 0$, $\Big($that is to say, when $\frac{\pi t}{T'} = \pi', 2\pi', 3\pi', \&c.\Big)$, $\cos \frac{\pi t}{T'} = \pm 1$, and the equation assigns to (θ) the successive values

$$\theta = \tfrac{1}{2} \Theta' (\mp \pi, 2\pi, 3\pi, \&c.)$$

That is to say, at each successive wave hollow and wave crest the range of the oscillation will be augmented by a definite amount of angle, namely, $\pi \times$ half the maximum slope of the wave, so that, but for friction of surface and keel resistance, a ship placed broadside to waves which have her own periodic time must ultimately roll completely over, however small the wave may be. Now, practically, it is not uncommon to find that the length of the half wave is 1-10th of the height of the wave, so that $\frac{H}{L} = 0{\cdot}1$, and $\frac{\pi H}{4L} = 0{\cdot}0785$, which treated as an arc, or angle, is half the maximum slope, or $4\frac{1}{2}^\circ$. Hence, with such waves at each successive roll, θ should increase by $(\pi \times 4{\cdot}5^\circ) = 14{\cdot}1^\circ$, and six successive steps, or three successive waves, should produce almost a complete overset; and though this conclusion, no doubt, requires to be limited on the score of many practical considerations (which will be noticed presently), it is obvious that in spite of all such limitations the concurrence of wave period and ship's period must produce the most formidable effects. Little modification is introduced into the result by assuming that the ship was in motion, or was inclined at an angle, when the first wave reached her. The terms which express the effect of these conditions (in the complete equation) follow, in this case, the same law of recurrence as that which governs the repetitions of wave impulse, and we may assume them to be such as will either diminish or increase the angle attained at any particular period; but since the deductions or additions are fixed in amount, while the angles due to wave impulse are increased continually, wave after wave, it is obvious that the same result must at last arise.

I may here state that the result is one which I have produced by direct experiment with floating bodies of extremely various forms; such as (1), a sphere immersed to two-thirds of its radius; (2), a prolate, spheroid, or egg-shaped body, immersed to about the same proportion of its major axis—the figure and the proportion having been arranged in relation to each other as to produce oscillations as near as might be isochronous, for large and small angles; (3), a body like a very flattened orange, wholly immersed, having only a very narrow neck which projected from it above the water level, serving like the stem of an hydrometer to regulate the depth of immersion. Each body was provided with ballast, having an adjustable level; and with a little care this was tentatively arranged so as to give to each the same natural periodic time in still water; though as No. 2 was isochronous for all angles, or nearly so, while Nos. 1 and 3 were sensibly slower in period for large angles, the identity of period could not be secured throughout. The bodies were placed in a trough, to which I had fitted an apparatus for generating a succession of waves of any required period, by taking the motion which created them from the crank axle of a fly-wheel, and driving the wheel by hand until the revolutions kept pace with the oscillations of an adjustable pendulum. This method was not, indeed, as exact as might be desired, but it did not admit of any very wide error; for the weight of the fly-wheel ensured an almost uniform speed of rotation, and the pendulum swinging before the eyes of the operator made it easy to maintain a general coincidence of speed. When then the oscillations of the floats and of the pendulum were made to synchronise, and the wave generator was run at the corresponding speed, all the floats were in turn overset after the transit of a very few waves. I say in turn, because the overset was not strictly simultaneous; but it was as nearly simultaneous as the somewhat rough character of the apparatus and mode of experiment warranted me in expecting, and it may be observed that at least so much accuracy was secured, that when the weight of either float was shifted only a very little, so as to make a very small change in its natural period, it was at once placed in a position plainly exceptional as compared with the others; and refusing to be completely overset by the series of waves which would upset the two others almost at the same moment, it was itself overset by a series slightly quickened or retarded, according as its own period was quickened or retarded by the altered position of its centre of gravity, while this slightly altered series of waves at once relieved the other two from their imminent peril.*

(2.) Another critical phase which the conditions of the equation of rolling motion may undergo, is when $\frac{T}{T'} = 0$, that is to say, when the ship may be supposed to have either infinite stability; or an infinitely small radius of gyration, or moment of inertia. This supposition refers to what is only mathematically assumable, not practically attainable; but it deserves to be noticed, with a view to the degree in which it may possibly be approached in practice.

If, then, we make $\frac{T}{T'} = 0$, all the terms of the equation vanish, except one.

$$\text{And } \theta = \frac{\pi}{2}\,\frac{H}{L}\,\frac{1}{1-o}\left(\sin\frac{\pi t}{T'}\right) = \frac{\pi}{2}\,\frac{H}{L}\,\sin\frac{\pi t}{T'}, \quad (12.)$$

which expression represents simply the slope of the wave. And, of course, this conclusion might have been anticipated: the ship, if she be perfectly quick in her movements, will follow precisely the slope of the wave; and probably the movements of a flat board laid flatwise on the water are a practical illustration of this condition. The periodic time of such a float may be practically treated as being = 0; and could a ship be so constructed as to fulfil this condition, as well as a flat board laid flatwise on the water fulfils it, there would be some wisdom in adopting the construction; for although the ship would be in the highest degree liable to disturbance, and would float conformably to the mean slope of every wave as it passed her, this very circumstance would limit her angles of inclination to the range of such angles as are practically found to exist in wave slopes; and these angles, in their most extreme development, are practically far within the limits which are reached by ships rolling under the effects of accumulated wave impulse. It would follow, too, from the principle insisted on as the basis of this discussion, that the mere angles of inclination occurring in this case, would create none of that disturbance which would at first thought be expected to follow from it; since it was shown that to a floating body, the wave surface where it floats is practically level, and this was exemplified by the experiment with the floating plumbline: indeed, disturbance would scarcely be felt, except in those parts of the ship which were raised high out of the water, and there it would be felt, not as due to the inclination, but as due to the translatory force developed by the *change* of inclination: and this force would be proportional to the distance of the point where the observer was placed from the ship's centre of gravity: were the observer stationed at the ship's centre of gravity, he would simply find that the changing inclinations of the deck placed it always in the position in which it would offer to him the firmest footing. But, in fact, it is impossible

* It had been intended to carry out a more complete series of experiments on the whole subject, and some progress had been made in improvements of the apparatus used; but an unavoidable change of residence involved the disarrangement of the whole, and I have never been able to undertake the reconstruction of it, much as I desire to do so.

to construct a ship which will even approximately fulfil the condition, for the practical necessities of construction assign to every ship a periodic time, at the least, so large as to be (speaking loosely) proportioned to her size; and if, unfortunately, misled by the term stability, the naval architect adopts a construction such that his ships shall have the smallest possible periodic time consistent with their size (and, *cœteris paribus*, this will be effected by giving to them the greatest possible stability), they will but thus the oftener meet waves with which they synchronise, and experience the consequence of the condition. But this point will be more fully treated near the end of the paper.

(3.) One other critical phase is deducible from the equation of rolling motion—or rather, the equation furnishes the exact conditions under which there will occur a phase, the existence of which, as a possible reality, is obvious on general principles; and, perhaps, these exact conditions will be accepted with the greater interest if the principles on which their existence depends are thus generally examined at the outset.

When the ship is rolling in still water, performing her excursions in her natural periodic time, it has already been shown that her angle of inclination at each instant is proportionate to the "sine of the time," dating the time from the moment when she is upright. But from the assumed nature of the wave curve, it follows that the angle of inclination of its surface also, is proportionate to the "sine of the time," dating the time from the point where it is level. Now, if when the ship is rolling in still water, we could alter her natural "effort of stability," increasing or diminishing it throughout in any given ratio, it would follow that the periodic time would be altered in the square root of that ratio; and an alteration equivalent to this may assumably be produced, if when the ship is in the act of rolling, and for a moment occupies a vertical position, a wave of appropriate dimension and period be made to approach her. The wave may be such, that if she is rolling towards it when it thus meets her, its growing inclination will precisely keep pace with the growth of her inclination, and will arrive at its maximum at the moment when her inclination, checked as it will have been (more rapidly than if she had remained in still water) by the increased force which the wave angle will have brought into operation, has attained its maximum. Then she will begin to recede towards the perpendicular (more rapidly than if in still water), being still urged by the increased force due to the wave angle; and since the point of maximum wave angle bisects the wave, it will follow that, as her outward quarter oscillation was exactly coincident, as regards time, with the first quarter of the entire wave, her homeward quarter inclination will be coincident, as regards time, with the second quarter of the wave; and this process will continue to be repeated as a steady oscillation of definite period, coincident with the period of the wave, but quickened as regards the ship's natural period in still water—quickened (that is) in a ratio, the square root of that in which the force due to the ship's inclination has been increased, by the addition of the force due to the wave angle. Conversely, if the ship be rolling from the wave instead of towards it, the force which would have checked her motion if in still water will be diminished at each angle of her inclination by the access of the wave angle;

and this may take place in such a manner and sequence as to produce a completely recurring oscillation, having now a period lengthened from that of the ship's natural roll to that of the slower wave, just as in the former case the period was quickened to that of a quicker wave.

The mathematical conditions by which these results may be determined, will appear on recurring to the general equation of motion (9), in which we must, in the first place, make $\alpha = 0$ in order to express the condition that the ship is upright at the commencement of the wave; and we must then equate to (0) the co-efficients of the remaining terms involving $\left(\frac{\pi t}{T}\right)$, that is to say, make

$$\frac{\pi}{2} \frac{H}{L} \frac{\frac{T}{T'}}{1 - \frac{T^2}{T'^2}} = \frac{U T}{\pi};$$

$$\text{or, } U = \frac{\pi^2}{2T'} \frac{H}{L} \frac{1}{1 - \frac{T^2}{T'^2}};$$

whence we have for the equation for (θ),

$$\theta = \frac{\pi}{2} \frac{H}{L} \frac{1}{1 - \frac{T^2}{T'^2}} \sin \frac{\pi t}{T'} \quad (13);$$

and if this be compared with equation (8), respecting still water oscillations, it will be seen to represent, precisely, a simple oscillation of the isochronous character; in which U, or the angular velocity of the ship when in the centre of the oscillation, where she is vertical, becomes

$$U = \frac{\pi}{T'} \frac{\pi}{2} \frac{H}{L} \frac{1}{1 - \frac{T^2}{T'^2}}$$

(an expression identical in value with that just now assigned to (U) for the ship's rolling motion under the assumed conditions), and in which the amplitude, or maximum angle of inclination, and which we may call (Θ), occurring when $\frac{t}{T'} = \frac{1}{2}$, becomes

$$\Theta = \frac{\pi}{2} \frac{H}{L} \frac{1}{1 - \frac{T^2}{T'^2}};$$

in which equation it may be recollected that $\frac{\pi}{2} \frac{H}{L}$ = the maximum slope of the wave, which, as before, we will call (Θ'): hence it will now be observed that the ratio of the ship's angle to the wave angle, or $\frac{\theta}{\theta'}$, maintains throughout the ratio of $\frac{\Theta}{\Theta'}$; the extreme point of the excursion happening at the middle part of the wave, and the ship being vertical at its top and bottom. Thus the period of the oscillation is identical with the period of the wave, and the whole circumstances are precisely those suggested by general considerations.

This general relation between the ship's angle and the wave angle being

$$\frac{\theta}{\theta'} = \frac{1}{1 - \frac{T^2}{T'^2}} = \frac{T'^2}{T'^2 - T'^2},$$

it follows that (θ) will be negative with respect to (θ') (that is, the angle of the ship will be in the direction opposite to the angle of the wave—or the masts will lean towards the wave) when T'^2 is greater than T^2; *vice versâ*, (θ) will be positive with respect to (θ') (or the ship's masts will slope in the same direction as the wave) when T'^2 is less than T^2.

In order to determine the limits of the relative value of T and T′, under which the form of oscillation is possible, solving for $\frac{T}{T'}$, we find $\frac{T}{T'} = \pm \sqrt{1 - \frac{\theta'}{\theta}} = \pm \sqrt{1 - \frac{\Theta'}{\Theta}}$, where we may dismiss the negative sign of the radical, as indicating a plainly irrelevant result.

The form of the expression shows that in the cases in which the ship has a quicker natural period than the waves, when, as was just observed $\frac{\theta'}{\theta}$ is positive (that is to say, when the ship, if oscillating thus, must be always inclined in the same direction as the wave, and must have her masts leaning from it, not towards it), the result becomes impossible when (θ) or (Θ) are less than (θ') or (Θ') respectively. The minimum possible value, therefore of $\frac{T}{T'}$ exists when $\frac{\theta'}{\theta}$ or $\frac{\Theta'}{\Theta} = 1$; and in this case it follows that $\frac{T}{T'}, = 0$, that is to say, the natural period of the ship must be infinitely short, or that of the wave infinitely long.

Both of these suppositions express what are only mathematically possible hypotheses; but in either case the result would be, that the masts of the ship must be at right angles to the wave surface, and the ship's oscillation must be identical with the wave oscillation. When, however, the value of $\frac{T}{T'}$ approaches this limit as closely as possible, the circumstances will be those of a very small vessel in very big waves. Thus, if we suppose a small yacht, having a natural period of one second, to be thus engaged with Atlantic rollers having a period of five seconds, and $\frac{H}{L} = 0{\cdot}1$, or $\Theta' = 0{\cdot}157 = 9°$, so that $\frac{T}{T'} = \frac{1}{5}$, it follows that

$$\frac{\Theta}{\Theta'} = \frac{1}{1 - \frac{1}{25}} = \frac{1}{0{\cdot}96} = 1{\cdot}04;$$

that is to say, her masts will be inclined in the same direction as the wave (*i.e.* will lean from the wave) at an angle 1·04 times the wave angle, or will be 4 per cent. more out of perpendicular than the wave is out of horizontal; so that with $\Theta' = 0{\cdot}157$ or 9°, Θ will $= 9{\cdot}36$.

If we refer to the other limit of the possible value of $\frac{\mathrm{T}}{\mathrm{T}'}$ when $\frac{\theta'}{\theta}$ or $\frac{\Theta'}{\Theta}$ is positive, this will be found when $\frac{\theta'}{\theta}$ or $\frac{\Theta'}{\Theta} = +0$, giving the value $\frac{\mathrm{T}}{\mathrm{T}'} = 1$. In this case it appears that the ship's inclination will be infinitely great with relation to the slope of the wave; and this result, though it expresses what, obviously, could never in fact happen (since a far smaller ratio of angle would be destructive, and what is more to the point, since the resistance of keel and surface friction render the supposition untenable), typifies, nevertheless, one form of the very critical results due to the condition that the natural period of the ship is closely concurrent with that of waves of large dimensions. It should be observed, that the dynamical interpretation of the theoretical result is, that the ship, when upright at the bottom of the wave, must have an infinitely great angular velocity in the direction of the wave motion; and that thus the force developed by the angles to which this velocity carries her will be infinitely great compared with that which the wave angle supplies; while the assumed property of isochronism determines the completion of the oscillation by help of the force thus developed within the period T′, which must thus = T.

If we turn, now, to the cases which arise on supposing $\frac{\theta'}{\theta}$ or $\frac{\Theta'}{\Theta}$ to be negative, when, as was pointed out, the ship's masts must lean towards the wave, and $\frac{\mathrm{T}}{\mathrm{T}'}$ is greater than 1, it follows that the possible values of $\frac{\mathrm{T}}{\mathrm{T}'}$ vary between 1 and $\propto$, those of $\frac{\theta'}{\theta}$ varying between -0, and $-\propto$.

In the former of these cases we have a result closely related to that just discussed, indeed identical with it, but that the infinite value of U, or the angular velocity with which the ship passes the upright position when at the bottom of the wave, *here*, is *towards* the wave, or in the opposite direction to the wave motion, *there*, was *from* the wave, or in the same direction as the wave motion; and the dynamical interpretation of the case is *mutatis mutandis* the same. The alternative limit, which gives $\frac{\mathrm{T}}{\mathrm{T}'} = \propto$ and $\frac{\theta'}{\theta} = -\propto$, implies that either the natural period of the ship is infinitely great, or that of the wave infinitely small, both of which suppositions represent what is a mere mathematical possibility. But the practical possibilities which approximately resemble them deserve notice.

If $\frac{\mathrm{T}}{\mathrm{T}'}$ be little in excess of 1 it will follow that when an oscillation of the character under discussion is established, the values of (Θ) must be extremely large, though the periods at which they recur will not differ very widely from T; if, for instance, we take the *Great Eastern*, or the *Duke of Wellington*, each of which has a period of about six seconds, and place the ship in waves of five and a half seconds, and of the same proportion as before, *i.e.*, with $\Theta' = 9^\circ$, then $\frac{T}{T'} = 1{\cdot}09$, and $\frac{T^2}{T'^2} = 1{\cdot}19$;

$$\therefore \quad \frac{\theta}{\theta'} \text{ or } \frac{\Theta}{\Theta'} = \frac{-1}{1 - \frac{T^2}{T'^2}} = -\frac{1}{\cdot 19} = 5{\cdot}28:$$

that is to say, on reaching the middle height and maximum slope of the wave, the ship will be inclined in the opposite direction, *i.e.*, her masts will lean towards the wave, with an angle 5·28 times as large = 47·5°.

Again, if $\frac{T}{T'}$ be much in excess of (1), the values of (Θ) will be small; and placing either of these ships in waves of the same proportion as before, but having a period of, say, two seconds, we shall have

$$\frac{T}{T'} = 3, \text{ and } \frac{\theta}{\theta'} \text{ or } \frac{\Theta}{\Theta'} = \frac{1}{1 - \frac{T^2}{T'^2}} = \frac{1}{\cdot 8889} = 1{\cdot}125;$$

that is to say, on reaching the middle height and maximum slope of the wave, the ship will be inclined with her masts towards the wave, at an angle = 1·125 times as large, or 10·125°.

And this latter aspect of the equation shows how it may happen, and I do not doubt that it does often happen, that a ship of slow period becomes engaged in oscillations of small amplitude with waves of a quick period, her movements becoming, in point of time, conformable to those of the wave.

And though I do not see that the equation will bring out the result (for, perhaps, the suppositions which render it possible are not included in the basis of the equation), it seems to me, on general principles, clear that some analogous oscillation might possibly obtain under some appropriate values of U and (α) (that is, angular velocity and angular position of the ship when at the bottom of the wave), such that the ship would roll right through one or more waves, and receive her impulse from the succeeding one, and would continue in this state of oscillation, engaged only with intermittent waves, as steadily as the equation shows she is capable of doing when engaged with every successive wave.

Moreover, since the special kind of oscillation which has just been discussed in relation to the equation rests on the supposition of the existence of a certain special value of U when the ship is at the bottom of the wave, coupled with the supposition that then (α) = 0, or that the ship is then upright: while, on the other hand, in tracing the motions derived from wave impulse solely, by help of the equation, though we proceed on the assumption that at the bottom of the first wave both U and (α) = 0, the equation assigns definite values to both U and (α) at the bottom of subsequent waves: it seems to follow, that an analysis of all the possibly-existing results derived from assumable values of U and (α) would lead to phases shading off by imperceptible degrees between such fixed oscillations as we have been recently considering, and those cumulative oscillations derived purely from wave impulse. And, lastly, a variety of results seem likely to follow from the supposition that the waves with which the ship is engaged, themselves pass through phases of maxima and minima, with varying proportions of length and height, while the equation has taken cognizance only of waves having unvarying period and unvarying dimension.

On this latter supposition it might ensue, that when exposed to growing waves, a ship engaged in what, with waves of fixed size and proportions, would be an oscillation of fixed magnitude, would undergo a series of increasing oscillations of the same character, each successive oscillation bearing to each successive wave very nearly the relation which would subsist between them were they both of fixed amount. This, therefore, points to a distinct type of cumulative oscillations.

Having thus discussed the special cases which the mathematical features of the equation suggest: (1), the case of perpetually-growing oscillation, due to recurring waves of period identical with that of the ship; (2), the case of non-oscillation, so to call it, when the ship's periodic time is so short, compared with that of the wave, that she must retain her position of equilibrium, keeping her masts perpetually at right angles to the wave; and (3), the class of cases under which it is possible that a ship may retain an oscillation of definite amount, when engaged with waves of definite period; together with some corollaries of this class; it becomes desirable to show the features which the equation develops, under the more ordinary conditions which it is capable of interpreting.

This may be better done by diagram than by mere numerical statements; and I refer the reader to the diagrams, Pl. XVIII., prepared in a manner suggested by my friend, Mr. Bell, which exhibits some of these features very intelligibly.

It was pointed out in reference to the general solution, that the result of the equation in the gross, was resolvable into two distinguishable elements: (1), the oscillations which would ensue were we to suppose the ship stationary and upright when a series of waves reached her; (2), the continued results of any oscillation which she might be in the act of undergoing when the waves reached her. And it was explained that the position of the ship at any moment was the combined result of the two operations;—*e. g.*, suppose that at the period of (n) seconds after the series of waves reached the ship, the equation of motion, applied on the supposition that she had been stationary and upright at the period when they reached her, would assign to her an inclination of 15° to starboard, and that, on the other hand, supposing that instead of being stationary and upright she had been undergoing an oscillation, which, irrespective of wave action, would assign to her, at the assumed period of (n) seconds, an inclination of 3° to port, it would follow that under the combined result of the original oscillation, and that derived from the waves, her true inclination would be $(15-3)^\circ = 12^\circ$ to starboard; had the position due to the original oscillation been 3° to starboard instead of 3° to port, then her true oscillation due to the combined result would be $(15+3)^\circ = 18$ to starboard, and so on.

It seemed best, therefore, to confine the diagrams to the elucidation of that part of the equation which exhibits separately the oscillation derived from the waves, partly because the step from this single result to the combined result is so simple, and partly because it is obvious, from practical considerations, that the results of the original oscillation must soon die out under the effects of surface friction and keel resistance.

The diagrams primarily represent what are the results of the equation applied to a ship supposed to have a period of 5 seconds (*i.e.*, to complete a single roll starboard to port, or *vice versâ* in that period), making, therefore, $T = 5''$; and assuming her to be exposed to several series of waves, having different periods or values of T'. The wave is in every case supposed to have a height from hollow to crest ten times it length from hollow to crest, or $\frac{H}{L} = 0{\cdot}1$, so that the maximum slope of the wave or $\Theta' = \frac{\pi}{2}\frac{H}{L} = 0{\cdot}157$, as assumed in former instances; but a mere alteration of vertical or horizontal scale will adapt the results to any other values of T or Θ'.

The results are shown in three figures. Fig. 1, gives the cases in which $\frac{T}{T'} = \frac{1}{1}$, $= \frac{4}{5}$ and $= \frac{5}{4}$; so as to exemplify the effect of waves having nearly the same period as the ship. Fig. 2 gives the cases in which $\frac{T}{T'} = \frac{2}{1}$ and $\frac{1}{2}$, the first case giving the wave only half, the second case twice the period of the ship. Fig. 3 gives the cases in which $\frac{T}{T'} = \frac{5}{9}$ and $= \frac{9}{5}$, showing the character of the difference which arises when the ratio of $\frac{1}{2}$ or $\frac{2}{1}$ is slightly varied.

For the more convenient reduction into diagram, the equation may be represented thus: call $\frac{T}{T'} = R$, and call $\frac{\pi}{2}\frac{H}{L} = \Theta'$. Then substituting and reducing,

$$\theta = \Theta' \frac{1}{\frac{1}{R} - R}\left(\frac{1}{R} \sin \frac{\sin R \pi t}{T} - \frac{\pi t}{T}\right).$$

In the diagrams, lengths along the abscissa represent time, and the corresponding ordinates represent the angles of inclination which the equation assigns to the ship at the corresponding periods. The base is divided into 5 seconds periods, in all the diagrams, corresponding with the assumed period of the ship; the vertical scale is the same in all; the diagrams therefore exhibit, in a form suited to ocular comparison and at a glance, what would happen to the assumed ship under the several conditions of wave period, supposing her motions to be governed by the equation. And it seemed that the comparison in this form would be rather more instructive than if based on the assumption that the wave period was the same in every case, and that the behaviour of ships of various periods was the subject of the enquiry.

The figures are constructed by laying off, on a base, two curves of sines corresponding with the two terms involved in the equation. The difference between the ordinates appropriate to any given instant of time, measured on each of the two curves of sines, becomes for the corresponding instant the ordinate of the curve which represents the angle of position. The uniform growth of maximum angle, in the case where $\frac{T}{T'} = 1$, is exhibited by the dark line curve in Fig. 1. The gradual accumulation and degradation of angle, in cases where the ratio $\frac{T}{T'}$ is somewhat greater or less than 1, as *e.g.*, with the ratio 5 : 4, or 4 : 5, is shown by the other two curves in the same figure, the dotted line belonging to the case in which T is greater than T′. Fig. 3 exhibits a curious feature, representing what may be called a baulked oscillation, in which the ship, in the progress of her return to an upright position, is met half-way by an approaching wave, and compelled to resume the extreme angle of inclination.

It will be seen that the character of each series of oscillations tends to make up a complete phase, at the end of which it simultaneously happens that for the moment the ship is both upright and at rest, so that in fact she has resumed all the conditions with which the phase commenced; and if the equation be studied, or if any one will take the trouble to follow the construction of the diagrams, it will be readily perceived that these conditions are fulfilled, and the phase begins again identically *de novo*, when if the ratio $\frac{T}{T'}$ be reduced to its lowest whole number terms, say $\frac{p}{q}$, $t = 2\,q\,T$. Thus, in Fig. 1, when $\frac{T}{T'} = \frac{4}{5}$ the phase runs to 10 T; *vice versâ*, when $\frac{T}{T'} = \frac{5}{4}$ the phase runs to 8 T; $\frac{T}{T'} = \frac{5}{9}$ gives 18 T; and $\frac{T}{T'} = \frac{9}{5}$ gives 10 T as the length of the phase, and so on.

It will not, I think, be disputed, that the results of the equation give, so far as character is concerned, a generally correct view of the phenomena of rolling motion. The cumulative, or baulked oscillations, and the recurring phases, serve, I think, to recall many not very agreeable recollections, to the mind or to the eyes of most persons who have experienced the rolling motion of a ship.

But quantitatively, it must be admitted, or rather it must be stated, that the angles of oscillation indicated by the equation are largely in excess of those which occur in practice; and it is necessary to point out the meaning of the difference, and to give some kind of rules, by which the real amount of inclination to be expected under any given circumstances, may be estimated. And as a preliminary step, it is advisable to remind the reader under what principal limitations and restrictions the solution has been pursued and obtained, though it is probable that most of these have been perceived and borne in mind, by any one who has intelligently followed the discussion, even when they have not been distinctly enunciated.

In the first place, irrespective of the fundamental defects of the assumed wave equation, it is very certain that a series of waves recurring at uniform intervals (*i.e.* with a strictly uniform value of T′) is not often to be with. Yet several waves often do occur in pretty regular succession, and it is on these occasions, that the greatest amount of rolling motion becomes developed: so that while, on the one hand, the attempt to obtain a perfectly general solution would be hopeless from its difficulty, turning as it would on many points which are entirely undetermined; on the other, it seems sufficient if we are able to determine what will happen under the most unfavourable circumstances which are in a general way to be expected.

Again, it has been throughout assumed, that the ship is strictly "in the trough of the "sea;" a position not often occupied wilfully. But, in the first place, the remark just now made as to the prudence of counting on the effect of the most unfavourable circumstances, is applicable here also. And, in the next place, while the assumption brings out the results in the most typical and characteristic form, it is easy to make an approximately true allowance for the effects of the oblique action of the waves; and it will be well to point out, in passing, a method of arriving at this.

The data from which the equation is deduced, are affected in two separate ways by the oblique action.

(1). If the ship is in motion, advancing towards the waves or receding from them, the periodic time of the waves is virtually accelerated or retarded by the circumstance; since the motion of the ship thus either shortens or lengthens the space which each successive wave has to travel in meeting her. And the acceleration, or retardation, is readily expressible in terms of the velocities of the ship and of the waves, and the angle of obliquity.

Thus, if (v) be the velocity of the ship, (v') that of the wave, and (ω) the angle of obliquity, or the want of parallelism, in the time T′ the ship will approach or recede from the waves by a space $= v\,\mathrm{T}' \sin \omega$, according as the angle leads towards, or from the waves. And in any other time (t), due to the space traversed by the ship towards or from the wave, the space will be $= v\,t \sin \omega$; so that $v\,(\mathrm{T}' \pm t) \sin \omega$ will be the actual space traversed by the ship from or towards the wave, during the altered wave period of $(\mathrm{T}' \pm t)$. This space, then, has to be added to or subtracted from L (the natural wave space) according as the ship moves from or towards the wave, in order to fix the distance which the wave has to travel in meeting the ship; but this space must also $= v'\,(\mathrm{T}' \pm t)$, so that $v'\,(\mathrm{T}' \pm t) = \mathrm{L} \pm v\,(\mathrm{T}' \pm t) \sin \omega$; and $(\mathrm{T}' \pm t)\,(v' \mp v \sin \omega) = \mathrm{L}$, $(\mathrm{T}' \pm t)$ being the amended virtual period of the waves.

$$\text{Hence } (\mathrm{T}' \pm t) = \frac{\mathrm{L}}{v' \mp v \sin \omega},$$

and this value will, therefore, have to be substituted for T′ in the equation of motion.

(2.) But, farther, the waves instead of acting each simultaneously on the whole length of the ship, as happens with direct waves, will act on it partially, or cornerwise, when approaching her obliquely; and the difference thus arising may be expressed approximately by assuming a smaller degree of steepness in the wave, if the assumption be duly governed by the circumstances; we must therefore find an appropriate new value of $\frac{\mathrm{H}}{\mathrm{L}}$ and substitute it in the equation of motion. The approximation may be arrived at as follows:—If we treat the ship as a parallel sided body, having her actual breadth (B), and a length (C), such as to give a rectangular area for the plane of flotation, equivalent to the real plane of flotation, the length (C) will, for ships of ordinary form, be about $\frac{2}{3}$, or $\frac{3}{4}$ of the total length of the ship. Now, if we observe the action of a direct wave when it begins to operate on our parallel sided ship, it is easy to see that for a given access of wave angle, the solid of wave-immersion will be nearly a parallel sided prism or wedge, having for its length the whole length of the ship (C), for its width half the beam, or $\left(\frac{\mathrm{B}}{2}\right)$, and for its depth say, (h). With the oblique wave, the solid of wave-immersion will affect only a corner of the body, and will be of a quasi pyramidal shape, having for its length $\left(\frac{\mathrm{B}}{2}\operatorname{cosec}\omega\right)$, its breadth $\left(\frac{\mathrm{B}}{2}\right)$, and its depth ($h$). So that the ratio of the former solid to the latter will be that of $\frac{1}{2}\,h\,\frac{\mathrm{B}}{2}\,\mathrm{C} : \frac{1}{3}\,h\,\frac{\mathrm{B}^2}{4}\operatorname{cosec}\omega$.

The ratio will, in fact, be $\frac{3\,\mathrm{C}}{\mathrm{B}\operatorname{cosec}\omega}$, in favour of the direct wave; only we must bear in mind, in reference to the steps by which the approximation was arrived at, that when ($\mathrm{B}\operatorname{cosec}\omega$) is greater than (C), the force assigned to the oblique wave is rather in excess; and that we must not ever count $\mathrm{B}^2\operatorname{cosec}\omega$ as greater than (3 C). Hence, instead of $\frac{\mathrm{H}}{\mathrm{L}}$ in the equation of motion, we may approximately put $\frac{\mathrm{H}}{\mathrm{L}}\,\frac{\mathrm{B}\operatorname{cosec}\omega}{3\,\mathrm{C}}$;

and the equation will become,

$$\theta = \frac{\pi}{2}\,\frac{\mathrm{H}}{\mathrm{L}}\,\frac{\mathrm{B}\operatorname{cosec}\omega}{3\,\mathrm{C}}\,\frac{1}{1-\dfrac{\mathrm{T}^2}{\left(\dfrac{\mathrm{L}}{v' \mp v\sin\omega}\right)^2}}\left\{\sin\frac{\pi t}{\left(\dfrac{\mathrm{L}}{v' \mp v\sin\omega}\right)} - \frac{\mathrm{T}}{\left(\dfrac{\mathrm{L}}{v' \mp v\sin\omega}\right)}\sin\frac{\pi t}{\mathrm{T}}\right\};$$

and this expression though not very compendious, does not contain more terms than necessary; and its reduction will be to some extent simplified by recollecting that v', or the wave's velocity $= \frac{\mathrm{L}}{\mathrm{T}'}$.

Thus treated, the equation gives results corresponding generally with the assumption of lower, or flatter waves—having a period quickened when the ship moves towards them, and retarded when she moves from them. The general reduction in the equivalent of wave angle corresponds with the general comparative steadiness and ease of any ship when quartering the waves; but the increase in the virtual wave period developed when the ship is receding from them, may readily bring a ship even of an extremely slow period into wave recurrences, which though individually weak, will in period nearly synchronise with her, and will produce angles of motion which, but for the principle of accumulation, would have been unaccountable. The point, however, does not seem to be of sufficiently serious consequence to deserve further consideration.

Lastly, the most important of the practical circumstances which have been disregarded in framing the equation, is the fact that the oscillations are performed in a resisting medium.

Now the best authorities have, I believe, ruled, that when a pendulum oscillates in a resisting medium, while the angle of excursion undergoes a gradual degradation the period remains unaltered. My own investigations, indeed, lead me to believe that this is not strictly true, except when the scale of resistance is small; but the period remains, at all events, so little changed, that so far as the results of the equation depend on T (the ship's period), their form will undergo little or no alteration in consequence of the fact that the medium in which she oscillates offers resistance to her motion. Indeed, when it is borne in mind that the distinctive character of the results, depends entirely on the degree in which the recurring wave impulses synchronise with the ship's motion, it is obvious that the periods of the phases which these motions undergo will remain very nearly the same, however much their magnitude may be limited by the resistance of the medium. And this view accords with the fact that in the results of the equation of motion, the character and period of the whole phase of oscillation in each case depends solely on the ratio $\frac{T}{T'}$, and is entirely unaffected by the steepness of the wave which governs, not the period, but the range or amplitude of the individual excursions. Hence, in attempting to measure the modifications which must be introduced into the conclusions derivable from the equation as it stands, by introducing the condition of resisting medium, we may safely use propositions based on identity of period, as instrumental to the determination of the effects of resistance. I cannot, indeed, pretend to give thus a direct and complete theoretical solution of the question, but I can point out how in the case of any individual ship an approximate solution may be obtained with certainty, by help of data derived from a single experiment with the ship herself, or with a carefully made model of her, tried in still water. And I believe that were a well-selected series of such experiments tried for ships of different forms, and the results tabulated, the series of corresponding solutions would enable us to determine, as if *a priori*, what modifications the results of our equation would require for any ship whatever.

Let us consider the nature of the facts which we observe, when any given ship or model is released and allowed to oscillate freely on still water, after having been drawn or forced into some angle of inclination. A number of oscillations ensue, each performed very nearly in the period T (accurately in it if the ship be isochronous), and each having a range somewhat smaller than its predecessor, until after a certain period the original effort is exhausted, and the motion ceases.

Now, the loss of range between any two consecutive excursions is a measure of the force expended in resistance during the intervening oscillation; and if the origin and character of the resistance be studied, it will be found that in any given ship having a given periodic time, the actual mean retarding force exerted during any oscillation is as the square of the range of the excursion; because in comparing one range with another, if each be divided into (n) equal parts, the isochronising law of force ensures that each unit of range in the one will be performed in the same period as the corresponding unit in the other; and since the length of each of these units is as the length of the whole range of which it forms a part, it follows that, unit for unit, the velocity throughout each range will be as the range, and therefore the resisting force, unit for unit, will be as the square of the range: and since the space traversed during the development of the force, unit for unit, is also as the range, it follows that, the "work done" (or force expended as distinguished from mere "force exerted," or intensity of force at any moment), will be as the cube of the range. But this is the resisting force which produces loss of range by producing loss of velocity; and we have to determine next, what loss of range thus corresponds with, or is due to, the expenditure of a given force or to a given amount of "work done," in resistance. If, with this view, we compare the dynamic value of any two oscillations of given range, we perceive that the "*vis viva*" in each (the "work done" stored up in each), is as the square of the whole range, because on comparing the effort of stability (that is, the force which urges the ship towards an upright position), for corresponding units of each range, we find that in each, the effort unit for unit, is as the whole range to which it belongs, since the isochronising law of force ensures that the force towards the position of rest, is throughout, as the distance from that position—that is to say, as the angle of inclination: and again, since here, also, the space traversed during the development of the force, unit for unit, is as the range, it follows that the dynamic value is as the square of the range.

Hence, we may express the losses of *vis viva*, or "work done" in resistance, as due to oscillations of various range, by the ordinates of a cubic parabola; the ranges being the measures of the abscissa due to each ordinate: while, similarly, the dynamic value of each range may be expressed by the ordinates of a common (*i.e.*, quadratic) parabola, based on the same values of abscissa. And if these two figures be duly related to each other in respect of constants, the successively decreasing ranges may be determined, by observing that they must step back along the base of the common parabola, by such gradations that the loss of ordinate due to each step, is the corresponding mean ordinate of the cubic parabola, or the ordinate of it measured at the central point of the step,—and it is very easy thus to determine the steps, by a simple bit of "tentative geometry." The method, it will be seen, may be simplified and reduced to a formula; but the reader will, I think, find the reduction more readily intelligible, if he will in the first instance follow the geometrical method. This is exemplified in Pl. XIX., Fig. 1. A B is a base line, measured off into equal spaces representing angles of excursion, or range of oscillation, *i.e.*, in terms of (Θ) on a scale of 5° to an inch. A C C is a cubic parabola, whose equation (selected at random), is $h = \cdot 000008\ \Theta^3$, and the ordinates of which are to show by scale the proportionate losses of range which will occur with various mean ranges or values of (Θ). To interpret the scale, let it be assumed as an experimental fact that the vessel, or model, has been pulled over to an angle of say 49°, and on being let go suddenly, has gone over in the opposite direction to 42°, making a loss of 7° of range on the oscillation. Place r_1 and r_2 at 49° and 42° accordingly, then $p_1\,h_1$, taken at 45·5° as the mean range of the oscillation, will be the mean ordinate of the cubic parabola for the two ranges; and representing therefore the loss of range due to a mean range of 45·5°, will be equivalent to 7°: hence the vertical scale will be $p_1\,h_1 = 7°$, and for an oscillation performed on any other mean range, say (Θ_n), the corresponding ordinate $p_n\,h_n$, if measured on that scale, will express the corresponding loss of range. A D D is a common parabola, whose proportions are determined as follows:—Draw $r_1 h' = p_1 h_1$ and join $r_2 h_1$, then the quadratic parabola must be such that in stepping along the abscissa from r_1 to r_2 its ordinate will be reduced by a quantity $= p_1\,h_1$; so that its tangent for the ordinate at 45·5° must be parallel to $r_2 h'$; but the tangent of a common parabola thus placed bisects the abscissa belonging to the point of contact. We, therefore, take P at $\frac{45 \cdot 5°}{2}$, and draw P Q parallel to $r_2 h'$, and then complete the parabola in the usual way. This being done, we can find the next step in the loss of range by tentatively ascertaining the point r_3, so that the difference between the quadratic ordinates at r_2 and r_3 shall be equal to the cubic ordinate at the middle position $p_2\,h_2$. And the process may be repeated accurately and rapidly for any number of successive oscillations.

But the arrangement admits of simplification as follows:—

The equations may manifestly be expressed in such terms, $h = a'\,\Theta^3$ and $h' = b'\,\Theta^2$, that (a') and (b') may stand for constant infinitesimals: and then the due relation of the two curves may be expressed in a quasi-differential form, as $\Delta h' = -\,a'\,\Theta^3$. Now by the quadratic equation $2'\,b\,\Theta\,\Delta\,\Theta = \Delta\,h$, so that $\Delta\,\Theta = -\,\frac{a'}{2b'}\,\Theta^2$ which is the equation of a common parabola, in which the ordinate $\Delta\,\Theta$ growing as the square of the abscissa, has to be used step by step, as a measure of the gradations by which the base must be marked off to show the successive reductions in the range of oscillation. This arrangement is shown in Fig. 2, in which A′ D′ D′ is the parabola, constructed of such a proportion that the loss of range in the first oscillation $r_1\,r_2$ is the height of the middle ordinate $p_1'\,h_1' = \cdot$, then it will follow that if $r_1\,r_2$ be taken $= p_2'\ h_2'$, it will be the next step of reduction in the series of oscillations, and so on, for $r_2 r_3$, $r_3 r_4$, &c. Thus $A r_1$ $A r_2$ .. $A r_n$ on the successive values of Θ_1 Θ_2 ... Θ_n, and this method would be sufficiently easy of application if we could always obtain as its basis the loss of arc in a single oscillation; but it often will occur that the loss of arc in (n) oscillations is more easily and certainly attainable, and then the method becomes less simple. Fortunately, however, we can reduce it into an equational form which will give, directly, the length of any required oscillation, if the loss of range in any other given number of oscillations has been first determined.

Thus, if we treat in a purely differential form, the equation last arrived at, and say $\frac{d\theta}{\theta^2} = p$, and then assume ($n$), the number of oscillations, to be fluxionably equicrescent, we may say $d\,n = q$.

$$\text{Hence,}\quad \frac{-d\theta}{\theta^2} = \frac{p}{q}\,d\,n;\ \text{then integrating}\ \frac{1}{\theta} = \frac{p}{q}n + \text{C};$$

And if by experiment, when $n = o$ (at the commencement of the oscillation), $\theta = \theta_o$, C will $= \frac{1}{\theta_o}$ and $\frac{1}{\theta} = \frac{p}{q}\,n + \frac{1}{\theta_o}$. Again, if when $n = m$ (*i.e.* after m oscillations), $\theta = \theta_m$, we have

$$\frac{1}{\theta_m} = \frac{p}{q}\,m + \frac{1}{\theta_o}\ :\ \text{hence}\ \frac{p}{q} = \frac{1}{m}\,\frac{\theta_o - \theta_m}{\theta_o\,\theta_m}\ \text{and}\ \frac{1}{\theta} = \frac{1}{\theta_o}\left(1 + \frac{n}{m}\,\frac{\theta_o - \theta_m}{\theta_m}\right),$$

$$\text{and generally}\ \theta_n = \frac{m\,\theta_o\,\theta_m}{(m-n)\,\theta_m + n\,\theta_o} = \frac{m\,\theta_o\,\theta_m}{n\,(\theta_o - \theta_m) + m\,\theta_m}\,. \quad (15.)$$

This equation expresses the successive ranges of oscillation, for any body oscillating by the isochronising law of force in a resisting medium, the law of resistance being that of the square of the velocity of the body's motion. It involves, indeed, the somewhat paradoxical conclusion that such a body will never come absolutely to rest, but so far as the loss of velocity depends on resistance varying as the square of the velocity, it is well known, and very easily proved, that the velocity is not absolutely destroyed in a finite time; and the fact that a body moving in a fluid does come absolutely to rest in a comparatively short period, only indicates that some other minor law of resistance is also in operation, which ceases to be insignificant, and indeed becomes responsible for the whole retardation, when the velocity has become extremely small. I will only add that results obtained from a twelve-foot pendulum having a four-inch iron shot for its "bob" oscillating in still water, accord very closely indeed with those indicated by the formula; and though these cannot be conclusively alleged in verification of it as applied to the oscillations of a floating body, they justify me in the hope that it will be found to answer sufficiently well there also if the experiment be carefully tried (which, however, I have not had an opportunity of trying since I found my way through the solution), and may be made to yield the corrections required for the results of the equation of rolling motion, so far as these depend on the operation of the resisting medium in which the ship rolls.

This, however, can be accomplished more readily by a tentative method applied to the diagram which represents the formula, than by using the formula itself. The chief practical use of the formula is as a help to the construction of the diagram. The principle of the application is as follows:—I assume in the first place that when a ship is performing an oscillation of given amplitude when rolling among waves, "the work done" in the shape of "resistance overcome," is the same as it would be were she performing an oscillation of the same range or amplitude in still water. No doubt the assumption is open to exception on minor points, especially if the angle of oscillation is for the moment inconsiderable when compared with the angle of the wave itself, but in its weightier phases I am satisfied it is a very fair approximation to the truth.

It will follow as a legitimate consequence of the assumption, that in balancing the account (so to speak) between "wave impulse" and "resistance," there must, in any given oscillation, be charged against growth of amplitude by impulse, as much loss by resistance as would have been exhibited in the shape of positive loss of range, had an oscillation of the same mean amplitude been performed simply in still water.

A single experiment performed on the ship herself, or a properly prepared model, in still water, will enable us to construct a diagram of resisted oscillations. This will be done by substituting for (θ) the values which experiment gives to (θ_o) and (θ_m) (the initial and the m h range of oscillation), and then deducing the value of (θ) for any two consecutive oscillations; these two will enable us to lay off the required parabola, as was done in Fig. 2, Plate XIX.; and then the ordinates of the parabola will show by mere measurement, for any value of (θ) the loss by resistance, which would occur were the ship oscillating with θ as its mean range.

The diagram should have for both vertical and horizontal scale that which was used as vertical scale in constructing the diagram of rolling motion; we can then take with the compasses from the one diagram the dimension which will serve as the proper correction for the other.

In order to make the operation here described more clearly intelligible, it shall be traced throughout, as it is performed in Fig. 3 and Fig. 4, Plate XIX., the former of which is the parabola showing loss of range by resistance, the latter a copy of the diagram showing the maximum phase of rolling motion, extracted from Fig. 1, Plate XVIII., only that in its horizontal scale, the unit of time (or the ship's natural period) is here treated as six instead of five seconds, in order to give the operation the practical character of showing the probable maximum roll of the *Great Eastern*, supposing her to fall in with a succession of storm waves, having also a period of six seconds. The assumed proportions of the waves are taken at 30 feet for the height, and 300 feet for the length (600 from crest to crest), which do not differ very widely from those of Scoresby's storm wave, as shown in Plate XVIII.

The parabola showing loss by resistance is based on an experiment tried with a model of the *Great Eastern*, of which I happen to have a record; and though its object was not that to which it will be now applied, so that for this purpose it is in several respects less complete and less definite than is desirable, it is sufficiently near the mark to render the result more interesting, and indeed more instructive, than if the operation had been based on a scale of resistances assumed at random.

In the experiment, the model was "hove down" to an angle of 45°, and then released. It had performed twenty-two well-defined oscillations, when the recoil undulations from the sides of the tank introduced some irregularity, and the "count" was stopped. I estimate the range of oscillation had then declined to about 2°. So that in the formula for θ_n (equation 15), we must put 45° for θ_o, 2° for θ_m, and 22 for m; and it thus becomes

$$\theta_n = \frac{22 \times 45^\circ \times 2^\circ}{n\,(45^\circ - 2^\circ) + 22 \times 2^\circ}$$

Then in order to determine the loss on the first oscillation, we make $n = 1 \therefore \theta_1 = \frac{1980^\circ}{87}$, or $22 \cdot 75^\circ$, so that in the first oscillation there is a loss of $22 \cdot 25^\circ$. Then since the parabola which is to furnish the scale of corrections for resistance based on the experiment, must be such that the ordinate for the mean of the values of (θ), with which the oscillation begins and ends, must equal the difference between those values; the mean value will in this case be $\frac{45^\circ + 22 \cdot 75^\circ}{2}$, or $33 \cdot 87^\circ$. So that the equation of the curve will be $h = \frac{22 \cdot 25^\circ}{(33 \cdot 87)^2}\,\theta^2 = \cdot 0194\,\theta^2$; and this is the parabola shown in Fig. 3, Plate XIX., the scale being the same as vertical scale in the diagram of rolling motion.

It will be recollected in reference to this diagram, Fig. 4, expressing the result of the equation of rolling motion, when $\frac{\mathrm{T}}{\mathrm{T}'} = 1$ and $\frac{\pi}{2}\,\frac{\mathrm{H}}{\mathrm{L}}$ (or the maximum slope of the wave) $= 9^\circ$, the non-resisted motion, under the successive synchronous wave impulses, produced a series of excursions of uniformly increasing range, the increase on each being $14 \cdot 1^\circ$.

Now on inspecting the scale of correction for resistance furnished by Fig. 3, it is seen that an oscillation performed on a mean range of $26 \cdot 9^\circ$, will experience in consequence of the resistance a loss of range $= 14 \cdot 1^\circ$; therefore, since each successive wave impulse is precisely capable of adding $14 \cdot 1^\circ$ to the range during each oscillation, it is at once clear that when the range has arrived at $26 \cdot 9^\circ$, the loss and the gain will precisely balance one another, and the oscillation will be continued with an unvarying range of that extent, so long as the period and figure of the waves remain unchanged. But we may trace step by step the gradations by which this condition is arrived at, as follows:—

It should be observed that each "mean range" is half the sum of any two consecutive ranges, so that if we set a pair of proportional compasses to the ratio of 1 : 2, we may sum up the two ranges in the longer legs, and the shorter legs will then give the mean range in such a form that if it be applied to the abscissa, in Fig. 3, the corresponding ordinate will be the loss of range due to resistance. The mean range finally taken for this purpose must be the corrected range, and it is for this reason that I have termed the method tentative; but anyone who will take the trouble to follow the operation, will find that it rapidly and certainly defines the required result. On referring to the diagram, it will be seen that the first oscillation proceeds from a range $= 0°$ in one direction, to a range $= 14{\cdot}1°$ in the other—let us say $= 0°$ to starboard, and $= 14{\cdot}1°$ to port—the mean oscillation, therefore, is $7{\cdot}05°$ prior to correction. The appropriate correction $a\,a'$ will readily be found, such as to correspond as ordinate, with a mean oscillation corrected by the deduction of $a\,a'$.

Then it would follow that if the next oscillation were performed without resistance, the ship would range to starboard $14{\cdot}1°$ further than this corrected range had taken her to port; and applying the sum of these ranges tentatively with the proportional compasses to the abscissa of the curve in Fig. 3, we can readily determine an ordinate $b\,b'$, such as to correspond with the mean oscillation corrected by the deduction of $b\,b'$; and if this process be repeated, step by step, we obtain the series of ranges indicated by the dotted line $a'\,b'\,c'\,d'\,e'\,f'\,g'$, which, it may be observed, almost immediately converges on what was shown to be the limiting angle of $26{\cdot}9°$.

The application of the correction for resistance is made the more simply and confidently in this particular phase of rolling motion, because the synchronism of the ship and the waves secures the condition that each oscillation is throughout performed "similarly" with relation to the position of the wave, so that each wave as it recurs supplies an identically repeated impulse to the ship. And if the experiment with the model of the *Great Eastern*, on which the corrected series is based, had been tried strictly with a view to the purpose to which it has been applied, I do not doubt that the result derived from it would have been a true representation of the movements which the ship would undergo under the assumed conditions, and which, as it stands, it represents only approximately. Again, were a similar experiment to be tried with the *Warrior*, or her model, and were the periodic time of the ship known, I feel sure that the same method of proceeding would enable us to predict truly how she would behave under similarly assumed conditions. In applying the method to those phases of rolling motion, in which the wave impulse, which is at the outset a cumulative, becomes by degrees an obstructive force, and where, consequently, the successive wave angles are not similarly placed in reference to the position and movements of the ship, more care and caution must be used; and though the broader features of the proper correction are readily discerned, it is not easy to produce a reliable representation of a completely corrected phase of motion.

In adverting to the experiments which it has been pointed out are required with a ship, or her model, in still water, in order to enable us to predict her behaviour among waves, a few words of explanation are required. With the ship herself, the running of a few men from side to side, will readily give her motion enough to enable an observer to determine her natural periodic time; while the shifting of a definite weight through a definite space, and an observation of the change of position which ensues, furnishes at once a direct measure of her stability; and this, when coupled with a knowledge of her total displacement, and of her metacentre (which are readily calculated from her lines and dimensions), and with her observed periodic time, give a measure of her radius of gyration, and of the position of her centre of gravity, both of which if deduced *â priori*, depend on laborious and intricate details of measurement and calculation. Again, the loss of range of oscillation, ensuing on a series of rolls beginning with a definite range, is easily determined by experiment and observation; for if a certain amount of oscillation in still water is set up by running the crew from side to side, and is then allowed to subside of itself, the men being kept in position, the series of ranges and the periodic time of each may be simultaneously noted.

For this purpose, however, the *primâ facie* indications of a pendulum-clinometer must not be accepted, unless it be suspended at the centre of gravity of the ship, because at any other point, the swaying motions of the position, due to its distance from the centre of gravity, of themselves produce a special deviation in the pendulum, on principles relied on in the early part of this paper; and these deviations become embodied in the deviation due to the inclined position of the ship, and thus confuse the indication and spoil its value. The amount of this confusion may indeed be calculated for still water, but in a seaway it defies calculation, owing to the bodily displacements which the ship herself undergoes, irrespective of change of inclination. In still water, the correction which the clinometer requires may be approximately stated in terms of its height above the centre of gravity. The principle on which the correction is made is as follows:—At the same level with, and close to the point of suspension, let a detached point be assumed to move horizontally, keeping pace, so far as horizontal motion is concerned, with the point of suspension, performing the reciprocation thus in conformity with the isochronising law of force in the time T (the ship's period); and if we take a common pendulum whose time is also T, and suspend it on a fixed point immediately over the position of rest of the pendulum attached to the ship, and make it oscillate in the same plane and with the same range as that of the horizontally moving detached point, it follows from the isochronising law of force, that all three points will exactly keep pace so far as horizontal motion is concerned, assuming, of course, that the range of the ship's oscillation is maintained.

Now were a plumb line, or short pendulum suspended from the detached point thus moving, it would at each instant hang parallel to the detached pendulum with which it keeps pace, instead of remaining upright; and for the same reason the clinometer will do the same, and the adventitious inclination thus created will at each instant form part of the whole angle which it indicates, and which is usually considered to be the ship's inclination; and if (h) be the height of its point of suspension above the ship's centre of gravity, and (l) the length of the detached pendulum, it follows from the identity of movement that approximately $h\,\theta = l\,\gamma$, (θ) being the ship's true inclination, and (γ) that of the detached pendulum, hence $\gamma = \frac{h}{l}\,\theta$: now ($\gamma$) is the angle to be deducted from the indication of the clinometer, which we may call (ϕ), to obtain the correct value of (θ), $\therefore\ \phi = \theta\left(1 + \frac{h}{l}\right)$, and $\theta = \phi\frac{l}{l+h}$. Now ($l$) must be such as to make the pendulum synchronise with the ship, $\therefore\ l = \mathrm{T}^2 \times$ second's pendulum $= \mathrm{T}^2 \times 3{\cdot}27$ feet, and $\theta = \phi\ \frac{l}{3{\cdot}27\,\mathrm{T}^2 + h}$. Thus, *e.g.*, in the *Great Eastern* where, $\mathrm{T} = 6''$, and for a clinometer suspended at the upper deck, $h = 30$, we have $l = 117$; and $\therefore\ \theta = \phi\ \frac{117}{117 + 30} = {\cdot}8\,\phi$: so that if the clinometer indicates $25°$, the true inclination $= {\cdot}8 \times 25° = 20°$.*

The only satisfactory method of noting angles of inclination which has suggested itself to me, and one which may be used with great facility and exactness, is that the observer should place himself amidships, on the upper deck, abreast of the main rigging, and at such a level as to be able to note the horizon on both sides when the ship is upright. He can then observe, when she is in motion, how many ratlines of the rigging she brings down to the horizon, port or starboard, and if his distance from the ratlines, their spaces, and the zero point, be duly noted in the first instance, the observations are readily reducible into measurements of angle.

It remains to draw attention to the practical conclusions which the results derived from the theory here presented, seem to suggest. First, as to the reliability of the theory itself: If it be said that the theory does not exhaust the subject, and that other conditions of paramount importance have to be taken into account, I answer that no doubt this is true to some extent; but if the theory is to be undervalued on that account, the extent to which those other conditions operate, ought to be placed on some tangible and measurable basis. Other causes, such as the stroke of a breaking sea, may occasionally carry a ship over with a lurch to some inclination quite independent of the results of this theory. But the occasional operation of such causes does not exempt a ship from the normal operation of the causes here elucidated. Some such cumulative action as has been described and investigated *must* occur; and its occurence, while it involves consequences too serious to be lightly regarded, suggests, by the principles on which it has been shown to depend, lines of thought in respect to those principles, by which remedial laws may be called into play.

The most important of these lines of thought is that which turns on the periodic time of the ship; and it seems impossible to resist the conclusion, that every ship should be made to have as slow a periodic time, as large a value of T as possible. It is true, that at the first glance, an adherence to this rule appears to involve dangerous results, from the circumstance that while it is plainly impossible to give to a comparatively small vessel

* Thus the chandeliers suspended from the saloon ceilings give very deceptive notions of the angles actually reached: though, nevertheless, they correctly show the attitude in which a man must stand to maintain his equilibrium.

a period larger than that of the largest possible waves, it follows that at some time or other she may fall in with waves having a period the same as her own; and that in proportion as her own period is large, the waves with which she synchronises will be of large dimensions; so that this trying coincidence, when it does occur, may be expected to assume a very serious form. And following out this train of thought, we might be led to infer that the safest course would be to give to each vessel the shortest possible period, so that it would be only with very small waves that she could synchronise.

But this mode of viewing the question involves several misconceptions.

In the first place, the practical necessities of construction and of stowage, render it impossible that any vessel should be made to synchronise with waves disproportionately small as compared with herself—that is to say, with waves so small that their effect on her would be limited on the score of mere dimension.*

In the next place, it is, I think, a matter of experience that small waves are steeper and more abrupt than large ones, when generated by wind of the same force.†

Now, it has been shown that it is not the absolute magnitude of the waves, but their steepness, which governs the scale of the rolling motions they will impress on a given ship. It follows, therefore, that she will oscillate through larger angles when synchronising with comparatively small waves than when with comparatively large ones. And as on the coming on of a gale the growth of the waves is undoubtedly gradual, if the period of the ship be slow, not only will the period of synchronism be longer deferred, but when it does occur its effects will be developed on a more moderate scale.

* To give this statement a somewhat quantitative character, I hazard the following tabular statement, the first two columns of which assign to vessels of various beam something like the periodic time, or value of T, which, under average circumstances, they will be found to possess. A third column shows the value of L, or length from hollow to crest of the ocean wave, which will have the same "period" as that of the vessel whose "beam" is figured in the same line.

Ship's Beam.	Ship's Period, or value of T.	Length of corresponding wave, or value of L, giving $T' = T$.
Feet.	$''$	Feet.
10	2·26	51
20	3·20	103
30	3·92	159
40	4·52	211
50	5·07	260
60	5·55	311

Probably, what are called "Symondite" ships, have, from their excessive stability, periods somewhat quicker than the table indicates; but this not in such a degree as at all to qualify the proposition to which the note refers.

† I believe the more formidable features of large waves are almost invariably due to the co-operation of the smaller waves, which, while the wind continues, always co-exist with large ones. A true breaker is very rare in large ocean waves; and such breakers as are occasionally met with among such waves, usually owe their existence to the circumstance that the combination of a larger and a smaller wave has locally created a crest or elevation, the dynamic conditions of which, if stated mathematically, would be found to render its resolution by any true wave motion impossible.

Moreover, since, as the gale continues, fresh and fresh waves continue to be formed and to grow, their growth being but little modified by the existence of their more full-grown brothers, the surfaces of the latter are interlaced with smaller waves of almost every intermediate stage of growth. Hence, it cannot be hoped that during the continuance of the gale a time will come when the waves have outgrown the period of a quickly oscillating ship.

And, again, though while the gale is subsiding, the smaller waves become more and more merged into the larger ones, or die out sooner from the failure of reproduction, so that the waves of longer period are also longer lived: yet these, themselves, not only subside rapidly in elevation, and, therefore, in slope, but must also lose some of their period; for, in truth, each wave, as it passes, is but another term of a series, the number of whose terms is inevitably limited by the position in which the waves commenced to be formed, and the successive terms of which are not only smaller in magnitude, but are also shorter in period, since they are nearer to the origin of their existence.

It must also be borne in mind, that those phases of rolling motion which are developed in the cases when $\frac{T}{T'}$ is a little greater than (1), run to maxima of less formidable extent than those which are developed when the ratio is reversed. It will be seen on referring to the diagram, that when $\frac{T}{T'} = \frac{5}{4}$, the extreme angles attained at the maximum period of the phase are considerably less than those which arise when $\frac{T}{T'} = \frac{4}{5}$; the ship is worse off when the period of the wave is a little in excess of her own, than *vise versâ*.

So that on the whole, it appears there are no circumstances under which material advantage can be practically gained by quickening the period of a ship, while there are many under which very material advantage can be gained by rendering it slow.

The idea sometimes occurs, that a good result would follow from giving a ship such a form that her period for large angles should differ very sensibly from her period for small ones; but it is less easy than might be expected, to produce this difference on so large a scale as would be required, within that compass of angles to which the rolling of a ship should be, if possible, limited; and such a partial application of the method as alone is within reach, might often lead to the very result which it was intended to guard against; because the ship might fall into a series of waves, the varying succession of which would just synchronise with that variation in her own period, which her increasing angles of inclination, under the successive wave impulses, would develop. And, remembering the almost proverbial manner in which wave phases culminate and then die out, how seldom they recur quite identically for any long period, it seems likely that a ship will oftener avoid continued synchronism by having an unchanging period—by being strictly isochronous—than by having a period which varies with her angles of inclination.

The surest and readiest method of giving a ship a long periodic time, is by lessening her stability under canvas. The effort of stability is the lever by which a wave forces a ship into motion—if a ship were destitute of this stability, no wave that the ocean produces would serve to put her in motion.

But though, of course, some such stability is essential, I believe that most large ships possess it in excess; *e.g.*, slow as is the period of the *Great Eastern*, I believe she could, with much advantage, part with a considerable portion of her stability. Let us see how the account stands :—When she is loaded to about 22,000 tons, her metacentre is about 8·7 feet above her centre of gravity, so that when she is forcibly inclined at an angle of 10° from her position of rest the effort of stability may be expressed by 22,000 tons $\times$ 8·7 $\times$ 10° = 22,000 $\times$ 1·52 = 33,400 $\times$ 1 = 334 $\times$ 100; that is to say, if a spar were rigged out from her side till its end was 100 feet from her centre line, 334 tons suspended on it would only incline her 10°. Now with the *Duke of Wellington*, only 50 tons suspended on such a lever would create the same inclination. Or if the measure be based on the ship's own dimensions, the *Great Eastern* would carry 800 tons at the extreme width of her own beam, with an inclination of 10°, while the *Duke of Wellington* will carry only 170 tons similarly placed with the same inclination. Yet the masts and spars of the *Great Eastern* are, in effect, scarcely double those of the *Duke of Wellington;* and the contrast will be intensified, and its bad results increased, by every step taken in the reduction of top hamper, which a mistaken policy or theory may suggest. But exclusive of masts and spars, on which, when bare, the effect of the wind is trivial (the strongest head wind when the sea is smooth scarcely taking five per cent. off her speed), the effect of a very strong wind on her long lofty side, ought to be taken account of. If she has stability enough to bear the stress of a hurricane (broadside on) on this, with a moderate inclination, no more need be required of her. Her side may be taken as 35 $\times$ 700 feet, and if we put 100lb. per foot as the force of a hurricane, the total pressure will be about 1,100 tons; and as the centre of effort for this force is about 30 feet above the centre of lateral resistance, the whole force may be considered as delivered at a leverage of 30 feet; and this, according to the scale of stability just given, would incline her only 9·87°; or if the ship were robbed of half her stability, were that possible, by stowing her cargo high, she would still bear the force of the hurricane on her broadside with an inclination of only 19·74.° But were her stability thus reduced, her period would be enlarged in the ratio of $\sqrt{2}:1$, and would be 8·5″ instead of 6″. And we have no reason to believe that waves of such period are ever formed.

The necessity of attending to the question of periodic time in reference to wave action is of far greater importance to large ships than to smaller ones: partly because in such ships the disuse of an external keel becomes almost a necessity, and although bilge pieces may be successfully substituted for a keel, as a means of enhancing resistance to rolling motion, and thus limiting the accumulation of angle, their application, if on a scale sufficient for the purpose, involves structural difficulty and practical inconvenience; so that thus such ships are the more capable of the development of cumulative rolling motion. And partly, because the larger the ship the greater is, *cœteris paribus,* the stress called into play by motions she impresses on the weights she carries.

These remarks, though they have been extended to an almost unreasonable length, cover but a small portion of the practical aspect of the question. I am conscious that they are at once lengthy and incomplete.

But, indeed, however imperfectly I have accomplished the task I have undertaken, I feel that the attempt has been justified by the importance of the subject, and by the substantial soundness and value of the views I have advanced; and, lastly, by the novelty of these views; for even if, contrary to my anticipations, it should prove that they have been held and taught elsewhere, in this country at least they seem to be unknown or ignored: and while in reference to structural strength, and forms best adapted for speed, our shipbuilders have been steadily adopting large improvements, based on scientific principles or on systematic experiments, in this particular branch of their art, they seem to have guided themselves either by rhetorical phrases or by random speculation: so that when a new ship is sent to sea, her constructor has to watch her behaviour in a sea-way with as anxious and uncertain an eye as if she were an animal he had bred and was rearing, and hoped would turn out well: not a work which he had himself completed, and whose performance he could predict, in virtue of the principles he had acted on in its design.

After the reading of the preceding Paper (in *abstract*), the following discussion took place*:—

Mr. J. Coryton: Allow me to suggest a mode of preventing the accumulation of oscillating momentum spoken of by Mr. Froude. The increase of that momentum is due, on the principle he lays down, to the fact that the centre of gravity being at a distance from an axis of rotation which may be taken as fixed, tends at each period of its displacement to produce about this axis a motion, the periodic time of which is nearly constant. The displacement is caused by external agency, usually that of the wind or the waves; and whenever, as it frequently happens in the latter case, the ship is struck at intervals of which the periodic time forms a multiple, the moment about this axis becomes increased. The conclusion—at first sight paradoxical—at which Mr. Froude arrives is, that the vessel oft he greatest stability is the one with no stability at all; and with vessels of the ordinary build that is undoubtedly the case. It is so, because the portions immersed in the act of rolling are so symmetrical as to cause little, if any, deviation of the axis of rolling.

My proposal is to obviate the co-operation of the successive external impacts, by a radical change in the form of ships, the outlines of which I mentioned at the meeting of last year. (See *Transactions I. N. A.*, Vol. 1, 1860, pp. 55, 56.) I hold in my hand a model of a vessel built upon my plan, which I am about to submit to the Trinity Board, in connection with the "Fairway system" of lighting chann els. You will

* It is necessary, in justice to the gentlemen who took part in the above discussion, for me again to remind the reader that a portion only of Mr. Froude's Paper was presented to the Institution in March last. Much of it appears here for the first time.—Ed.

see that, in a system of construction such as this—where, in fact, the forward horizontal sections are approximately those of a vertical wedge, and the after vertical sections also approximately those of a horizontal wedge—this accumulation of momentum is impossible. On the occurrence of a slight lateral disturbance in such vessels, the axis about which motion takes place becomes changed, owing to the irregularity of the figure immersed, to such an extent that any external impetus will inevitably tend rather to neutralise than augment the momentum under the influence of which the body is in motion.

Mr. J. Scott Russell, Vice-President: I think it is our duty to return our thanks to Mr. Froude for the very luminous account he has given us of his investigations. For a long time since, I have known that he was engaged in interesting experiments, and we have been fortunate in inducing him to come out of his retirement to give us the benefit of them. I hope this will form an important paper in the forthcoming *Transactions*, in which we shall have opportunity for considering at our leisure all the reasons for his belief. Some of the things he states seem improbable enough, and some appear probable enough.

We are indebted for some of the most interesting investigations on this subject to our learned and reverend chairman, who will, I trust, favour us with some remarks on the present occasion.

I think, with regard to a good deal which Mr. Froude has said, that the sense of the matter is this,—that a piece of solid matter which floats in wave water, really does in motion that which we know it does in rest, that is to say, that a solid body floating in water is subject to all the pressure which the water in the space it occupies would be subject to under the same circumstances. What Mr. Froude says about the pendulum amounts to this,—that in wave motion a floating body obeys laws very closely approximating to, if not identical with, the laws which water would obey in the place which the body occupies. If we understand wave motion pretty well, and I fancy we do, then ordinary wave motion takes place in cycloidal lines, and we know that the relations of these cycloidal lines give this as the reading of them, namely, that every particle of water occupies the particular place it does, merely because it is obeying the original impulse which disturbed it, and at the same time the action of gravity. Therefore, that Mr. Froude's float should keep its place at right angles to the wave in the manner it does, is merely a proof that for the time being he has succeeded in putting a solid body in the same conditions as the fluid around it, and that the solid body may be taken as subject to the law of wave motion.

With regard to the existence of two independent motions of the body, that is interesting, because the same sort of law which governs the rolling of a body, if it be approximately circular, governs wave motion also. If that is so, the oscillations of the body, and of the water, are affected by the coincidences of the two waves, and these would lead exactly to *maxima* at definite intervals, and to *minima* at other definite intervals. What inference we are to draw from this, in regard to the peculiar construction of ships with reference to rolling, is, however, matter open to discussion. What I apprehend the sailor would choose is, that when he finds himself in a difficulty with his ship broadside on to the sea, he will put her end on; for he has the choice of two oscillations, and can choose the one, and reject the other. But there can be no doubt about this, that the easily rolling vessel is generally the vessel which has the least stability, if by a stable vessel you mean the vessel in which the action of the water upon the ship has the greatest tendency to preserve the masts of the ship at right angles to the surface of the water. What we call a crank ship, or tender ship, is generally a very easy ship. A tender ship has generally, but not inevitably, little stability, because, though a vessel may be of a shape which will not have a powerful tendency to keep her masts at right angles to the troubled waters, she may have stability given to her by the disposition of her weight. A tender ship, having little tendency to remain at right angles to the water, but having only that measure of stability which you can regulate by the disposition of the weights, unquestionably may be made an easy ship.

In regard to the construction of the iron-plated ships, there is no doubt that this question of stability and rolling has become a most important one, and it will now require the most exact investigation which such men as our learned chairman, and Dr. Woolley, and Mr. Froude, can bring to bear upon it, to put us in a position to handle it effectually.

The CHAIRMAN (the Rev. Canon Moseley, M.A., F.R.S., Vice-President): Having myself, a long time ago, entered upon some mathematical investigations connected with this subject of the rolling of ships, I may be permitted to bear my testimony to my sense of the great value of the communication which has now been laid before the Institution. I hardly recollect ever listening to any scientific paper which seemed to me more fruitful of results, or more likely to suggest ideas which may hereafter fructify. Most especially I think the subject of the synchronism of the oscillation of waves and of vessels, which Mr. Froude has brought before us—which synchronism has been written upon before, but has never before been investigated in the way in which he has investigated it,—I say, I think this has an important bearing upon the construction of ships; and the elaborate scientific attention given to it by him will be of inestimable value in the theory of shipbuilding. I allude particularly to the way in which the repetition of the impulses exerted upon the body is made to modify the oscillations. If, when a vessel first rolls, and then rolls back, at the instant it has got to the extreme roll back, you apply a second impulse, you very considerably increase the amount of the succeeding oscillation; and if you wait till the third extreme roll, and then apply a third impulse, you impel her through a still further angle; and so you may go on, applying such impulses at those particular periods. The effect of such accumulation of motion is displayed in the case of suspension bridges, some of which have been brought down by the steady marching of a body of troops, owing to the fact of the steps of the troops synchronising with the oscillations of the bridge. The view of the subject which has been taken by Mr. Froude, is one that could only be taken by a person who is familiar with mathematical questions of this kind, and it shows that a great harvest of valuable results is to be reaped from the mathematical discussion of the more profound problems of shipbuilding science. Whether the action of water upon a ship is like the action of water upon a little float is, I think, a question which admits of considerable doubt. The relation there may be between the motion of a great ship, in reference to the waves which rock it, and that of the little float, mentioned by Mr. Froude, is a subject to which I would direct the attention of those who are interested in these matters.

There is this to be borne in mind. In discussing the question of the stability of a ship as dependent upon the position of the centre of gravity of the ship, we must remember that the rolling depends partly upon the centre of gravity of immersion. We may measure the work required to incline a ship from the upright position down upon her side by the wind or the waves. This work may be measured by calculating the number of pounds lifted one foot high which would be requisite to lift the whole ship bodily up through a space equal to the difference between the heights through which the centre of gravity of immersion, and the centre of gravity of the ship itself, would respectively rise during the inclination. Take the difference between the elevations of the two centres, then the work required to lift the ship through a height equal to that, is the work that must be done upon the side of the ship to bring her over to the proper angle. That is going upon the supposition that the ship is in still water, and that the water remains still when she has been deflected, and is left to return. The great merit of this investigation is, that it takes into account the influences of the fluid in which the vessel is floating. But this is to be borne in mind—that the expression of the time of oscillation has reference to the *stability* of the ship, as well as to this *work*. To what extent these two elements separately influence the motion of the ship remains for further discussion.

I would add, that the influence of the motion communicated to the fluid in the act of rolling was perfectly apparent in the experiments made at Portsmouth by Mr. Fincham. Experiments were made with models of circular sections and triangular sections. With the circular, the body moved the water round it but very little, and in that respect theory agreed with experiment; but when we took up the triangular model, then the water was displaced by the triangle of immersion, and there was great discrepancy between the actual results of experiments and the results of theory.

To illustrate Mr Froude's paper on the Rolling of Ships.

Fig. 1.

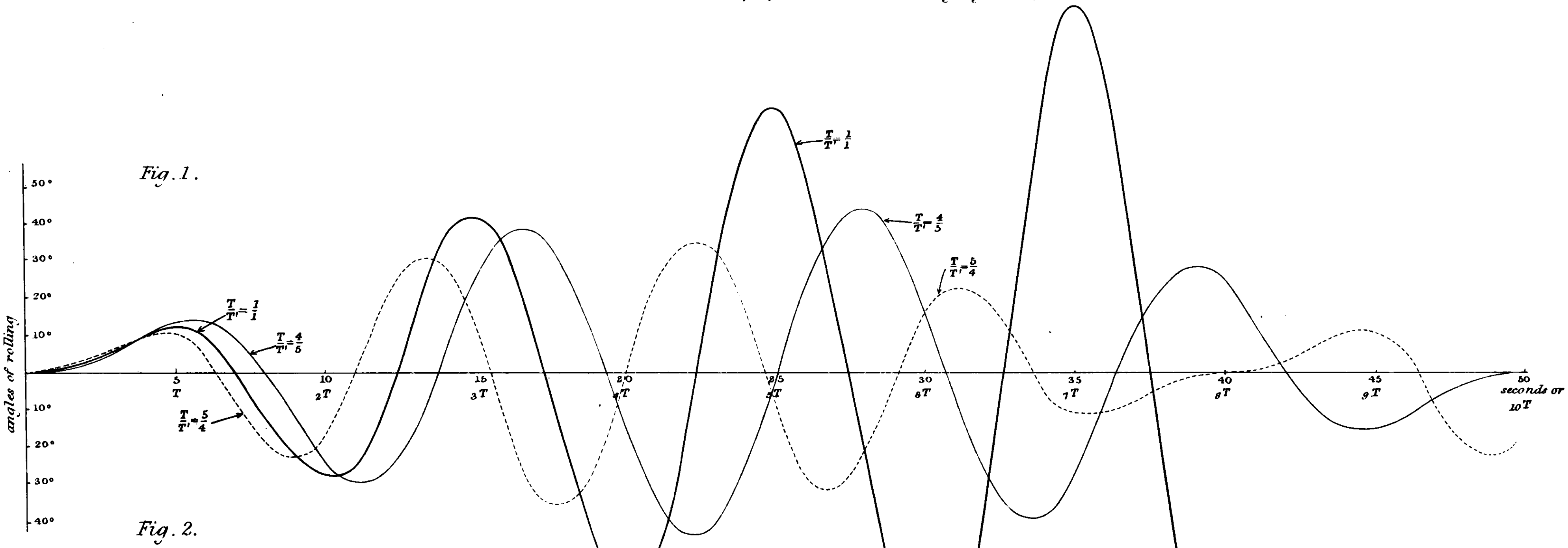


Fig. 2.

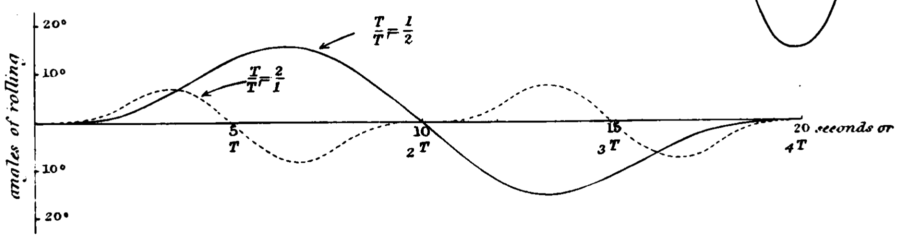


Fig. 3.

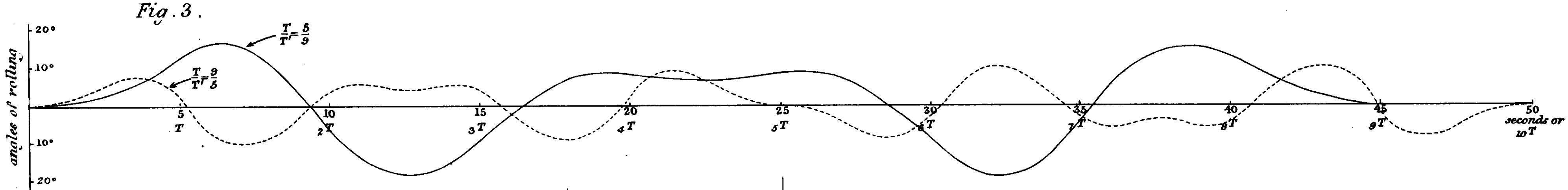
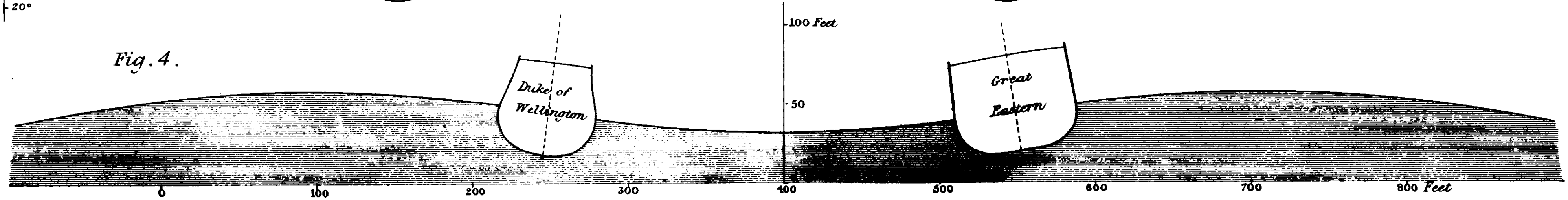


Fig. 4.



Atlantic storm wave observed by Dr Scoresby, March 6. 1848. – $\frac{\pi}{2}\frac{H}{L}$ = 136 or 7°8

To illustrate Mr. Froude's paper on the Rolling of Ships.